Superconductivity - M1

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TD 7: The Cooper problem

1 Electron-electron interaction mediated by phonons

An important contribution to the understanding of superconductivity was made by Cooper, who, in 1956, recognised that the ground state of an electron gas is unstable if one adds a weak attractive interaction between each pair of electrons. Such an interaction had been discussed by Fröhlich in the form of a phonon-mediated interaction. The strength of the effective interaction due to exchange of phonons depends on the detailed structure of the metal. In practice it can be strong enough to win over the repulsive Coulomb interaction (in some frequency range). We can model this interaction as a weak constant attraction in a shell of extension $\hbar\omega_D$ above the Fermi surface, with ω_D the Debye frequency.

Consider the ground state of the non-interacting Fermi gas (all states with $E < E_F$ filled and with $E > E_F$ empty). Now we add two electrons in states just above E_F. A weak attractive interaction is switched on in the form of phonon exchange. Due to phonon exchange the two additional electrons continually change their wave vector, whereby momentum must be conserved: $\mathbf{K} = \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}'_1 + \mathbf{k}'_2$. Knowing scattering of pairs lowers the potential energy, which pair momentum $\mathbf{K}$ leads to the lowest potential energy?

2 Cooper pairing and instability of the Fermi sea

At first sight the attractive effective interaction seems to be too weak in order to bind a pair. Indeed, in 3D two particles must interact with a minimum strength in order to form a bound pair. However, Cooper argued that the presence of the $N-2$ other electrons in the Fermi sea combined with the Pauli principle radically alters the two-electron problem so that a bound state exists no matter how weak the attraction!

Let us study two fermions at temperature T = 0 on the top of the Fermi sea, interacting with each other but not with the other fermions. This is known as the Cooper problem.

1) Give the Schrödinger equation associated with the two-particle wave function $\psi(\mathbf{r}_1, \mathbf{r}_2)$ and interaction potential $\hat{V}_{int}(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2)$ for an energy E.

In the non-interacting case $V_{int} = 0$, the wave function of the two-particle state $(\mathbf{k}, -\mathbf{k})$ is

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{V}} e^{i\mathbf{k}_1 . \mathbf{r}_1} \frac{1}{\sqrt{V}} e^{i\mathbf{k}_2 . \mathbf{r}_2} = \frac{1}{V} e^{i\mathbf{k}.(\mathbf{r}_1 - \mathbf{r}_2)} \ . \tag{1}$$

Similarly, the most general representation of a two-particle state with center-of-mass momentum $\mathbf{K} = \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{0}$ and a non-zero interaction $V_{int} \neq 0$ is given by the series:

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{V} \sum_{\mathbf{k}} \chi(\mathbf{k}) e^{i\mathbf{k}.(\mathbf{r}_1 - \mathbf{r}_2)} \ . \tag{2}$$

2) Show that the Schrödinger equation reduces to:

$$\Big(2\epsilon(\mathbf{k}') - E\Big)\chi(\mathbf{k}') = -\frac{1}{V} \sum_{\mathbf{k}} V_{int}(\mathbf{k}' - \mathbf{k})\chi(\mathbf{k}), \tag{3}$$

with $\epsilon(\mathbf{k}) = \hbar^2 k^2/(2m)$.

3) What are the constraints on the probability $|\chi(\mathbf{k})|^2$ of finding the pair with wave vectors $(\mathbf{k}, -\mathbf{k})$?

As briefly discussed in the previous part, the potential is attractive for wave vector pairs separated by an energy of the order of the Debye energy $|\epsilon(\mathbf{k}) - \epsilon(\mathbf{k}')| < \hbar\omega_D$. For simplification, we assume the interaction potential to be constant within a thin layer around the Fermi surface:

$$V_{int}(\mathbf{k}' - \mathbf{k}) = -V_0 \quad \text{for } E_F < \big(\epsilon(\mathbf{k}), \epsilon(\mathbf{k}')\big) < E_F + \hbar\omega_D \tag{4}$$

$$= 0 \quad \text{otherwise} \tag{5}$$

where $V_0 > 0$ and $\hbar\omega_D \ll E_F$.

4) Use a self-consistent reasoning to show that:

$$1 = \frac{V_0}{V} \sum_{\substack{\mathbf{k} \\ E_F < \epsilon(\mathbf{k}) \\ \epsilon(\mathbf{k}) < E_F + \hbar\omega_D}} \frac{1}{2\epsilon(\mathbf{k}) - E} \tag{6}$$

5) Prove that a two-electron bound state exists, whose energy is lower than that of the fully occupied Fermi sea ($T = 0$) by an amount

$$2\Delta = E - 2E_F = \frac{2\hbar\omega_D}{1 - e^{\frac{2}{V_0 g(E_F)}}} \approx -2\hbar\omega_D e^{\frac{-2}{V_0 g(E_F)}} < 0 \ , \tag{7}$$

with $g(E_F)$ the density of states for one spin type at the Fermi energy.

This bound state is called a Cooper pair. Explain why the ground state of the non-interacting free electron gas becomes unstable when a weak attractive interaction between electrons is "switched on". We have only considered the problem of one Cooper pair. In reality the instability leads to the formation of a high density of such electron pairs via which the system tries to achieve a new lower-energy ground state. This new ground state is the superconducting state.