Superconductivity - M1

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TD 6: The Josephson effect

The Josephson effect is an important manifestation of macroscopic quantum coherence and was predicted by Brian Josephson in 1962 while a 22-year-old graduate student. The effect is observed when two superconductors are connected by a so-called weak link (prominent examples are tunnelling barriers, point contacts or normal conducting layers connecting the two superconducting electrodes). Such a contact is called a Josephson junction.

Josephson junctions are classified as either lumped or extended, depending on whether the spatial extension of the contact is taken into account or not, i.e., whether the phase and current vary across the junction. Lumped or zero-dimensional Josephson junctions have a spatially homogeneous supercurrent density and phase difference. Extended Josephson junctions in turn are classified as either short or long, depending on whether their spatial dimension is smaller or bigger than a characteristic length scale named *Josephson penetration depth* λ_J (see below).

We wish to study the impact of a magnetic field on the junction. Remind that the supercurrent density is given by

$$\mathbf{J}_s = \frac{e_*}{m_*} n_s^*(\mathbf{r},t) \left[\hbar \nabla\theta(\mathbf{r},t) - e_* \mathbf{A}\right] \ . \tag{1}$$

The current density is gauge invariant [1], and therefore we introduce a gauge-invariant phase gradient [2]

$$\gamma = \nabla\theta(\mathbf{r},t) - \frac{e_*}{\hbar}\mathbf{A} \ . \tag{2}$$

The density $|\psi|^2 = n_s^*$ is typically much smaller in the junction than in the superconducting electrodes. Moreover, since $\mathbf{J}_s$ is the same in the electrodes as in the junction area (by conservation of current), γ is negligibly small in the electrodes compared to the junction region (see Fig. 1). We can then replace the gauge-invariant phase gradient γ by the *gauge-invariant phase difference* Θ given by

$$\Theta(\mathbf{r},t) = \int_1^2 \gamma(\mathbf{r},t) = \theta_2(\mathbf{r},t) - \theta_1(\mathbf{r},t) - \frac{2\pi}{\Phi_0}\int_1^2 \mathbf{A}(\mathbf{r},t)\cdot d\mathbf{l} \ , \tag{3}$$

where the integration path is taken along the direction of the current and $\Phi_0 = e_*/h$. That is, the path across the junction from superconductor 1 with phase θ_1 to superconductor 2 with phase θ_2 .

The first Josephson relation or current-phase relation is given by

$$J_s(\Theta) = J_c \sin\Theta \ , \tag{4}$$

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[1] We recall that the electric potential V_E and the magnetic vector potential $\mathbf{A}$ determine $\mathbf{E}$ and $\mathbf{B}$ -fields via $\mathbf{E} = -\nabla V_E - \frac{\partial \mathbf{A}}{\partial t}$ and $\mathbf{B} = \nabla\times\mathbf{A}$. The EM fields remain unaffected under the gauge transformation $\mathbf{A}\to\mathbf{A}' = \mathbf{A}+\nabla f$ and $V_E \to V_E' = V_E - \frac{\partial f}{\partial t}$ with $f(\mathbf{r},t)$ a differentiable function of $\mathbf{r}$ and t . In order for the Schrödinger equation for a q -charged particle in an EM field to remain unaffected, the wave function ψ should be concurrently subjected to the phase transformation $\psi(\mathbf{r},t)\to\psi'(\mathbf{r},t) = \psi(\mathbf{r},t)\ e^{i(q/\hbar)f(\mathbf{r},t)}$.

[2] Note that $\nabla\theta(\mathbf{r},t) - \frac{e_*}{\hbar}\mathbf{A}$ cannot be written as $\nabla\gamma$, that is, as the gradient of a gauge invariant phase, since this would imply $\nabla\times\mathbf{A} = \mathbf{B} \equiv 0$.

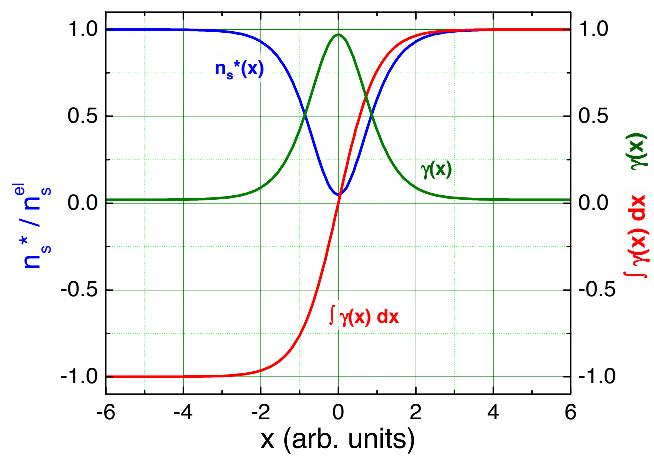

Figure 1: Sketch of the variation of the superconducting density n_s^* and the gauge-invariant phase gradient γ across a one-dimensional Josephson junction extending in the x -direction. The integral $\int \gamma dx$ is also shown. The figure illustrates that γ is only changing considerably in the junction region, while varying negligibly in the superconducting electrodes, justifying the introduction of the gauge-invariant phase difference Θ across the junction.

with J_c the critical or maximal current density. The second Josephson relation or voltage-phase relation is given by

$$\frac{\partial \Theta}{\partial t} = \frac{2\pi}{\Phi_0} \int_1^2 \mathbf{E}(\mathbf{r}, t) \cdot d\mathbf{l} \tag{5}$$

where the line integral on the r.h.s corresponds to a voltage drop across the junction. In parts 1 and 2 we will work in the stationary state at zero voltage ($\frac{\partial \Theta}{\partial t} = 0$).

1 Short Josephson junctions in presence of magnetic flux

Consider the setup shown in Fig. 2. Two superconductors of spatial dimensions L_x, L_y, L_z are connected by a weak link of thickness d in the x -direction. We assume $L_y, L_z \gg d$ so that effects of the edges of the junction can be neglected. The thickness of the electrodes L_x is assumed to be much larger than the London penetration depths λ_1 and λ_2 of the two superconductors. A magnetic field of strength H_0 is applied is the y -direction.

1. Give the magnetic field $B(x)\mathbf{e}_y$. Sketch its profile. In which direction do the screening currents run?

2. Give the flux Φ through the contour in Fig. 2 as a function of $H_0, d, \Delta z, \lambda_1, \lambda_2$, where λ_i are the London penetration depths of the two superconductors. Take $\Delta x \gg \lambda_1, \lambda_2, d$. What is the effective length of the gap, t_B , also called the magnetic thickness?

3. Let $\Theta(P) - \Theta(Q)$ be the shift of the gauge-invariant phase difference between two positions P and Q along the z -axis separated by a distance Δz . Find the relation between this shift and the flux. To this end, integrate the phase gradient $\nabla\theta$ along a closed path perpendicular to the field (dashed line in Fig. 2), where we take $\Delta x \gg t_B$ and Δz infinitesimal. Split the integral

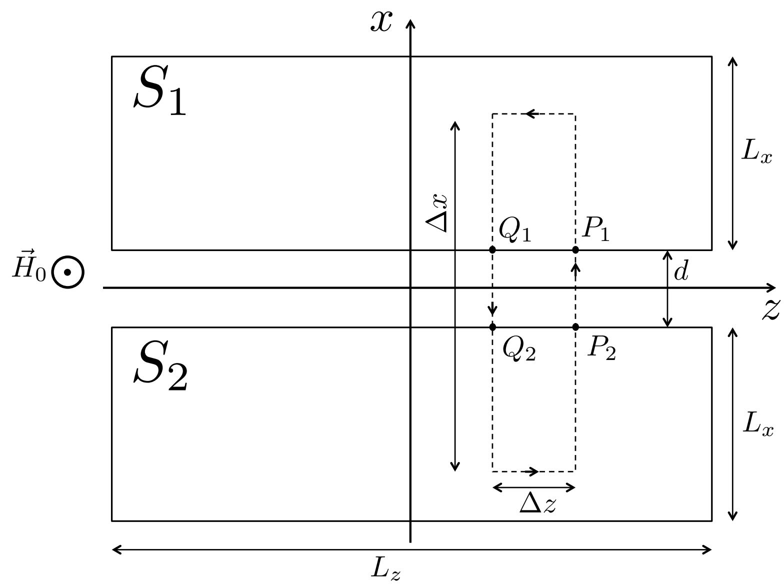

Figure 2: A Josephson junction: two superconductors coupled by a weak link (of width d). The weak link could be a thin insulating barrier or a short section of non-superconducting metal.

into convenient segments. Since the junction is short, we neglect the magnetic field generated by the Josephson current itself (self-field) – we will justify this later.

4. Use the current-phase relation to obtain the current density. What is its oscillation period? What is the magnetic flux going through one such period?

5. How does the total current $I_s = \int\int J_s(y,z)dydz$ across the junction depend on the magnetic field? Assume that the maximal Josephon current density $J_c(y,z)$ is spatially homogeneous in the junction. Give the maximal Josephson current I_s^m as a function of the total magnetic flux Φ through the junction. Make a figure.

6. Draw the Josephson current density $J_s(z)$ for the cases $\Phi = 0$, $\Phi = \frac{1}{2}\Phi_0$, $\Phi = \Phi_0$ and $\Phi = \frac{3}{2}\Phi_0$.

2 Long Josephson junctions in presence of magnetic flux

We now consider a long Josephson junction, where the spatial dimensions of the junction are bigger than the Josephson penetration depth. We still use the setup shown in Fig. 2, and consider a situation where the field and the current does not depend on time. We assume $J_c(y,z) = const$ and that the current flows in the negative x -direction.

1. When the effect of the induced self-fields cannot be neglected, show that

$$\frac{\partial^2\Theta}{\partial z^2} = \frac{1}{\lambda_J^2}\sin\Theta \, . \tag{6}$$

Give the expression for λ_J . This nonlinear differential equation is called the 1D *stationary Sine-Gordon equation.*

2. Why is the Josephson penetration depth a penetration depth? To answer this question, linearize the above equation for small phases and get an expression for $B_y(z)$. Show that by taking into account the self-fields, for weak applied fields, the currents will tend to screen the field from the interior of the junction.

3. Explain why the short Josephson junction ($L_z \ll \lambda_J$) is equivalent to neglecting the self-fields.

4. Show that
$$\Theta(z) = 4\arctan(e^{\frac{z-z_0}{\lambda_J}}) \tag{7}$$
is a particular solution of the stationary Sine-Gordon equation for an infinitely long junction with boundary conditions $\Theta(z) \to 0$ and $\Theta'(z) \to 0$ for $z \to \pm\infty$. Calculate the corresponding magnetic field and the Josephson current density. What are the total current and the total flux? Make a figure. This solution is known as a soliton or fluxon or Josephson vortex.

3 Shapiro steps

In this section we wish to study AC circuits. Let us consider a circuit driven by an AC voltage. A practical way to realise such a modulation is to subject the Josephson junction to microwave radiation.

Let us apply both a DC and AC voltage across the junction:
$$V = V_0 + V_1\cos(\omega t)\ . \tag{8}$$

1. Determine the phase difference and the supercurrent across the junction.

2. Use the expansion
$$e^{iz\sin\alpha} = J_0(z) + 2\sum_{k=1}^{\infty} J_{2k}(z)\cos(2k\alpha) + 2i\sum_{k=0}^{\infty} J_{2k+1}(z)\sin((2k+1)\alpha) \tag{9}$$
and the parity $J_k(z) = (-1)^k J_{-k}(z)$ of the Bessel functions to show that
$$I_s = I_c\sum_{k=-\infty}^{+\infty}(-1)^k J_k\left(\frac{e_* V_1}{\hbar\omega}\right)\sin(\phi_0 + \omega_J t - k\omega t)\ . \tag{10}$$
Determine ω_J .

3. Add the shunt current V_0/R and determine the DC part of the total current $I = I_s + V_0/R$. Plot I as a function of V.

Realistic circuits are driven by a current. Applying the AC current in the RCSJ model allows to determine the voltage as function of the current. The current-voltage curves then show so-called Shapiro steps. The jumps occur precisely when the average voltage $\langle V\rangle = k\frac{\hbar\omega}{e_*}$ for $k = 0, \pm1, \pm2, \ldots$