## Problem 1 Symmetric group

1. Let $G$ be a group and $S_N$ the symmetric group, i.e., the group of permutations of $X = \{1, \ldots, N\}$. Show that any group homomorphism $\varphi : G \to S_N$ induces a group action on $X$ by the action $g \cdot x = [\varphi(g)](x)$.
2. Consider $\sigma \in S_N$ and the action of $\mathbb{Z}$ defined by the homomorphism

$$\begin{aligned} \varphi_\sigma : \mathbb{Z} &\to S_N \\ j &\mapsto \sigma^j, \end{aligned}$$

where we interpret negative powers as powers of the inverse permutation. Find the orbits of this group action for the permutations

| $j$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\sigma_1(j)$ | 3 | 1 | 2 | 4 |

| $j$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\sigma_2(j)$ | 3 | 4 | 1 | 2 |

| $j$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\sigma_3(j)$ | 4 | 2 | 1 | 3 |

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3. Which of the permutations $\sigma_1$, $\sigma_2$, $\sigma_3$ are in the same conjugacy class?
4. A cycle is defined by a $\mathbb{Z}$-orbit of a permutation with the elements written out in the order in which they occur. Any permutation is uniquely characterized by its cycles. Write out $\sigma_1, \sigma_2, \sigma_3$ and their inverses in terms of their cycles (cycles consisting of a single element are omitted). Use the cycle notation to calculate $\sigma_1 \circ \sigma_2$, $\sigma_2 \circ \sigma_1$, $\sigma_3 \circ \sigma_1^{-1} \circ \sigma_2$. What is the order of a permutation that can be expressed as a $k$-cycle, i.e., a single cycle of $k$ elements and $N - k$ cycles of length one? Show that every $k$-cycle with $k \geq 2$ can be written as a product of $k-1$ (not necessarily disjoint) 2-cycles.
5. Let $\sigma, \tau \in S_N$ with $\sigma = (j_1, \ldots, j_k)$ a $k$-cycle. Show that $\tau\sigma\tau^{-1} = (\tau(j_1), \ldots, \tau(j_k))$.
6. The cycle type of a permutation is given by the lengths of all of the cycles it contains. Prove that the conjugation classes of $S_N$ are defined by the cycle type.
7. Express each of the conjugation classes of $S_4$ by a Young diagram, following this recipe: Draw each $k$-cycle as a row of $k$ squares, all cycles stacked on top of each other with larger cycles on the top, aligned on the left. Convince yourself that each diagram represents a partition of 4. Identify the diagrams describing $\sigma_1$, $\sigma_2$, and $\sigma_3$. Find examples of permutations for the other diagrams. How many elements does each conjugacy class have?
8. Consider a cube centered at the origin. Assign four different labels to the eight vertices, putting the same label onto pairs of vertices at $p$ and $-p$. Study the action on these labels for a rotation around a midpoint of a surface, a vortex, and a midpoint of an edge. Show that $S_4 \simeq O$.

## Problem 2 Representations

1. Show that irreducible representations $r$ over a $\mathbb{C}$-vector space $\mathcal{E}$ must have the property that $r[g] = \lambda \mathrm{Id}_{\mathcal{E}}$ with $\lambda \in \mathbb{C}$ for all $g$ from the center $Z(G)$ of the group $G$.

2. Show that two one-dimensional irreducible representations are equivalent if and only if they are the same.
3. Find all irreducible representations of $\mathbb{Z}/n\mathbb{Z}$ over $\mathbb{C}$. Hint: Consider the action of the cyclic group on the complex unit circle.