# ICFP M2 - Statistical physics 2 – TD n° 3
## The mean-field $p$-spin glass model

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In this TD we shall study with the replica method the thermodynamics of the fully connected $p$-spin glass model, defined by its Hamiltonian

$$H(\underline{\sigma};\underline{J}) = - \sum_{1 \le i_1 < i_2 < \dots i_p \le N} J_{i_1 i_2 \dots i_p} \sigma_{i_1} \dots \sigma_{i_p} \ . \qquad (1)$$

The Ising spins $\sigma_i$ have $p$-body interactions ($p = 2$ corresponds to the Sherrington-Kirkpatrick model), the coupling constants $J_{i_1 \dots i_p}$ are Gaussian i.i.d. random variables of zero mean and variance $\frac{p!}{2N^{p-1}}$. We denote $\mathbb{E}[\bullet]$ the average over these random couplings.

We will be mostly interested here in the case $p \ge 3$ ; even though these multi-body interactions do not seem microscopically motivated, the properties of this model has strong similarities with the ones of the structural glasses, and a mean-field theory for the glasses, called Random First Order Transition, was built starting from the $p$-spin model. Moreover this type of interaction appears naturally in the interdisciplinary applications to computer science.

1. Show that the energies $H(\underline{\sigma};\underline{J})$ are correlated Gaussian random variables with zero mean and covariance
$$\mathbb{E}[H(\underline{\sigma};\underline{J})H(\underline{\tau};\underline{J})] = N\frac{1}{2}q(\underline{\sigma},\underline{\tau})^p(1+o(1)) \qquad (2)$$
when $N \to \infty$, where $q(\underline{\sigma},\underline{\tau}) = \frac{1}{N}\sum_{i=1}^N \sigma_i \tau_i$ is the overlap between the two configurations.

2. Explain why this model should become equivalent to the random energy model in the limit $p \to \infty$ (taken after the thermodynamic limit $N \to \infty$).

3. Compute the annealed free-energy $f_{\rm a}(\beta)$ of the $p$-spin model.

The computation made during the lectures showed that, when $n$ is a positive integer,

$$\lim_{N\to\infty} \frac{1}{N} \ln \mathbb{E}[Z(\beta,\underline{J})^n] = \sup_Q A(Q) \ , \qquad \text{with} \quad A(Q) = n\frac{\beta^2}{4} + \frac{\beta^2}{4}\sum_{a\neq b} q_{ab}^p + S(Q) \ , \qquad (3)$$

where $Q = \{q_{ab}\}$ is an $n \times n$ matrix, with 1 on the diagonal, encoding the overlaps between the $n$ replicas of the system and $S(Q)$ the entropy of such configurations. The latter term can be computed to obtain

$$A(Q) = n\frac{\beta^2}{4} + n\ln 2 - \frac{\beta^2}{4}(p-1)\sum_{a\neq b} q_{ab}^p + \ln\left( \frac{1}{2^n} \sum_{\sigma^1,\cdots,\sigma^n} \exp\left[ \frac{\beta^2}{4} p \sum_{a\neq b} q_{ab}^{p-1} \sigma^a \sigma^b \right] \right) \ . \qquad (4)$$

To determine the quenched free-energy we want to use the replica trick and express

$$f_{\rm q}(\beta) = -\frac{1}{\beta} \lim_{n\to 0} \frac{1}{n} A(Q_*) \ , \qquad (5)$$

where $Q_*$ is the saddle-point dominating $A$. To take the limit $n \to 0$ we have to make an ansatz on the form of $Q$, as we shall now discuss.

4. We start with the simplest and most natural Replica Symmetric (RS) form of the matrix $Q$, with $q_{ab} = q \geq 0$ for all $a \neq b$.

   (a) Show that such a saddle-point yields the following free-energy,

$$f_{\rm RS}(q;\beta) = -\frac{\beta}{4} - \frac{1}{\beta}\ln 2 + \frac{\beta}{4}(pq^{p-1} - (p-1)q^p) - \frac{1}{\beta}\int_{-\infty}^{\infty} \frac{{\rm d}z}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \ln\cosh\left(\beta\sqrt{\frac{pq^{p-1}}{2}}z\right) \ ;$$

   to perform this computation you should use the identity

$$\int_{-\infty}^{\infty} \frac{{\rm d}z}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2 + az} = e^{\frac{1}{2}a^2} \ . \qquad (6)$$

   (b) Check that $f_{\rm RS}(q=0;\beta) = f_{\rm a}(\beta)$, the annealed free-energy.

   (c) Analyze the behavior of the various terms of $f_{\rm RS}$ in the limit $q \to 0$, and conclude that $q = 0$ is a local maximum for $p \geq 3$.

   (d) The best estimate of the quenched free-energy within the RS ansatz is $f_{\rm RS}(\beta) = \sup_{q\in[0,1]} f_{\rm RS}(q;\beta)$ (the maximization instead of the usual minimization being a counter-intuitive consquence of the $n \to 0$ limit). Assuming that $q = 0$ is the global maximum, argue that a phase transition must occur for some $\beta \leq 2\sqrt{\ln 2}$ (you should compute the entropy associated to $f_{\rm RS}$).

5. This phase transition manifests itself as Replica Symmetry Breaking (RSB) of the relevant saddle-point $Q_*$. The simplest way to break the symmetry between the $n$ replicas is to divide them into $n/m$ groups of $m$ replicas, and to take $q_{ab} = q_1$ for the off-diagonal elements of the $n/m$ diagonal blocks of the matrix $Q$, i.e. when $a \neq b$ are two distinct replicas of the same group, and $q_{ab} = q_0$ in the off-diagonal blocks, i.e. when $a$ and $b$ are in different groups. This is the first level of replica symmetry breaking (1RSB). In the present model one can actually take $q_0 = 0$, as suggested by the RS solution.

   (a) Compute $A(Q)$ for such a matrix, and take the limit $n \to 0$ with $m \in [0,1]$ to obtain

$$\begin{aligned} f_{\rm 1RSB}(q_1,m;\beta) &= -\frac{\beta}{4} - \frac{1}{\beta}\ln 2 - (1-m)\frac{\beta}{4}(p-1)q_1^p \\ &+ \frac{\beta}{4}pq_1^{p-1} - \frac{1}{\beta m}\ln\left[\int_{-\infty}^{\infty} \frac{{\rm d}z}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \cosh^m\left(\beta\sqrt{\frac{pq_1^{p-1}}{2}}z\right)\right] \ . \end{aligned}$$

   (b) How does this expression simplifies when $q_1 = 0$ ? and when $m = 1$ ?

The estimate of the quenched free-energy of the model at the 1RSB level is obtained by maximizing $f_{\rm 1RSB}(q_1,m;\beta)$ with respect to $q_1$ and $m$, both in the interval $[0,1]$. The equations obtained by imposing the stationarity of $f_{\rm 1RSB}$ with respect to $q_1$ and $m$ have different type of solutions depending on the temperature :

- At high temperature, above a temperature that we call $T_{\rm d}$, there is only one solution corresponding to $q_1 = 0$. In this regime we recover the high temperature (replica symmetric) solution discussed previously. The temperature $T_{\rm d}$ depends on $p$: for $p = 3$ one finds $T_{\rm d} \simeq 0.681598$, whereas for $p \to \infty$ one has $T_{\rm d} \to \infty$.

- Below $T_{\rm d}$ and above a temperature that we call $T_{\rm c}$ one finds two solutions: (1) the high temperature solution discussed before, (2) a new one corresponding to $m = 1$ and $q_1 > 0$. As it can be easily checked, they have the same free energy. The temperature $T_{\rm c}$ depends on $p$: for $p = 3$ one finds $T_{\rm c} \simeq 0.651385$, whereas for $p \to \infty$ one finds $T_{\rm c} \to \frac{1}{2\sqrt{\ln 2}}$, which is the critical temperature of the random energy model.

- Below $T_{\rm c}$ the optimal solution corresponds to $0 < m < 1$ and $q_1 > 0$. The high temperature solution exists at any temperature, as discussed previously, but is not optimal for $T < T_{\rm c}$.

The interpretation of these transitions will be discussed in the next lecture.