ICFP M2 - Statistical physics 1 – TD n° 1
Martin-Siggia-Rose formalism for the Ornstein-Uhlenbeck process: response and correlation

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In this tutorial, we study the Ornstein-Uhlenbeck (OU) process which is a one-dimensional stochastic process $x(t)$ that evolves via the Langevin equation

$$\dot{x}(t) = -\mu\, x(t) + h(t) + \zeta(t) \;, \tag{1}$$

starting from an initial position (at time t_0) $x(t_0) = x_0$. In Eq. (1), $\mu > 0$, $h(t)$ is a (small) external field which is used here to probe the response of the process and $\zeta(t)$ is a Gaussian white (thermal) noise with zero mean, i.e. $\langle \zeta(t) \rangle = 0$ and $\langle \zeta(t)\zeta(t') \rangle = 2T\delta(t-t')$ where $T > 0$ is the temperature. For $\mu = 0$, one recovers the standard Brownian motion. The OU process was initially introduced to describe the velocity $x(t) \equiv v(t)$ of a massive Brownian particle under the influence of friction. Since then, the OU process has become a fundamental stochastic process to describe noisy relaxation in various situations. As such, it has found various applications in statistical mechanics but also in financial mathematics (e.g. to model interest rates), biology (e.g. to model neuronal activity) or computer science (e.g. in queuing theory).

The objective of this tutorial is to compute, from the Martin-Siggia-Rose (MSR) action corresponding to the OU process, the two-time correlation $C(t,t')$

$$C(t,t') = \langle x(t)x(t') \rangle \Big|_{h=0} \tag{2}$$

and the response function $R(t,t')$

$$R(t,t') = \left\langle \frac{\delta x(t)}{\delta h(t')} \right\rangle \Big|_{h=0} \tag{3}$$

where in Eqs. (2) and (3) the notation $\langle \cdots \rangle$ denotes an average over the thermal noise ζ.

1 A direct approach without MSR

In fact, the equation of motion (1) is exactly solvable. This exact solution, which we derive here, will serve as a guideline for later computations with MSR.

1. Compute explicitly $x(t)$, the solution of Eq. (1) at time t, for a single realization of the noise $\zeta(t')$, for $t_0 \leq t' \leq t$.
2. From this explicit solution compute the correlation $C(t,t')$ and the response $R(t,t')$, for a fixed initial condition x_0.
3. In the limit $t_0 \to -\infty$ show that these averaged observables are: i) independent of x_0, ii) stationary (i.e. time-translation invariant) with $C(t,t') \to C_{\rm st}(t-t')$ and $R(t,t') \to R_{\rm st}(t-t')$ and iii) that they are related by the relation

$$R_{\rm st}(\tau) = -\frac{1}{T}\frac{dC_{\rm st}(\tau)}{d\tau} \;, \quad \text{for } \tau \geq 0 \;, \tag{4}$$

   which is called the "Fluctuation Dissipation Theorem" (FDT). What about the existence of this FDT regime in the Brownian limit $\mu \to 0$?

2 Computation of $C(t,t')$ and $R(t,t')$ using the MSR action

We consider the limit $t_0 \to -\infty$ where we have seen that the correlation and response are independent of x_0. We thus set $x_0 = 0$. We recall that the MSR action corresponding to the stochastic equation that governs the OU process in (1) on the time interval $(-\infty, +\infty)$ with $h = 0$, denoted as $S_{\rm OU}[x,\hat x] \equiv S_{\rm OU}[\{x(t'),\hat x(t')\}, -\infty < t' < +\infty]$, reads

$$S_{\rm OU}[x,x'] = \int_{-\infty}^{+\infty} dt'\, i\hat x(t') \left[\dot x(t') + \mu\, x(t')\right] + T \int_{-\infty}^{+\infty} dt'\, \hat x^2(t')\ , \tag{5}$$

where $x(t')$ and $\hat x(t')$ are real scalar fields, such that $x(t' \to \pm\infty) = \hat x(t' \to \pm\infty) = 0$.

1. It is thus useful to introduce a two-component vector $\phi(t)$

$$\phi(t) = \begin{pmatrix} x(t) \\ i\hat x(t) \end{pmatrix} \tag{6}$$

such that $S_{\rm OU}$ in Eq. (5) can be written as

$$S_{\rm OU} = \frac{1}{2}\phi : G^{-1} : \phi = \frac{1}{2}\int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2\, \phi(t_1) G^{-1}(t_1,t_2)\phi(t_2)\ . \tag{7}$$

Show that G^{-1} has the following block structure

$$G^{-1}(t,t') = \begin{pmatrix} 0 & \delta(t-t')\,(-\partial_{t'} + \mu) \\ \delta(t-t')\,(\partial_{t'} + \mu) & -2T\delta(t-t') \end{pmatrix} \tag{8}$$

2. Show that the matrix $G(t,t')$ which encodes the correlation and response has also a block structure

$$G(t,t') = \begin{pmatrix} C(t,t') & R(t,t') \\ R(t',t) & 0 \end{pmatrix} \tag{9}$$

where $C(t,t')$ and $R(t,t')$ are solutions of

$$\partial_t R(t,t') + \mu\, R(t,t') = \delta(t-t') \tag{10}$$
$$\partial_t C(t,t') + \mu\, C(t,t') = 2T R(t',t)\ . \tag{11}$$

3. Show that the stationary solution $C(t,t') = C_{\rm st}(t-t')$, $R(t,t') = R_{\rm st}(t-t')$ found above in 1.c) is solution of this coupled set of equations (10) and (11).

3 An alternative computation via the generating function

We introduce $Z[J,\hat J] \equiv Z[\{J(t'),\hat J(t')\}, -\infty < t' < +\infty]$, the generating function, defined as

$$Z[J,\hat J] = \int \mathcal{D}x\mathcal{D}\,\hat x\; e^{-S[x,\hat x] + i\int_{-\infty}^{+\infty} dt'\, J(t')x(t') + i\int_{-\infty}^{+\infty} dt'\, \hat J(t')\hat x(t')}\ . \tag{12}$$

As we show here, $Z[J,\hat J]$ can be evaluated explicitly for the OU process, which then allows to extract $C(t,t')$ and $R(t,t')$.

1. Show that the correlation $C(t,t')$ in Eq. (2) and response $R(t,t')$ in Eq. (3) are obtained from $Z[J,\hat J]$ as

$$C(t,t') = -\frac{\delta^2 Z}{\delta J(t')\delta J(t)} \tag{13}$$

$$R(t,t') = -i\frac{\delta^2 Z}{\delta J(t)\delta \hat J(t')} \tag{14}$$

2. Compute explicitly the generating function $Z(J, \hat{J})$ and show that it can be written as

$$Z[J, \hat{J}] = e^{-\int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 \left(\frac{T}{2\mu} e^{-\mu |t_1 - t_2|} J(t_1) J(t_2) + i\, \theta(t_2 - t_1)\, e^{-\mu (t_2 - t_1)} \hat{J}(t_1) J(t_2) \right)} \tag{15}$$

Hint: observe first that the (path) integral over the field x in Eq. (12) can be performed explicitly, using a generalization of the identity $\int_{-\infty}^{\infty} e^{iq\,y} dq = 2\pi\delta(y)$.

3. Compute explicitly $C(t, t')$ and $R(t, t')$ from $Z[J, \hat{J}]$ and check that you recover the expression of $C_{\rm st}(\tau)$ and $R_{\rm st}(\tau)$ found above.

4. (*Optional*) We now consider t_0 finite. Compute $Z[J, \hat{J}]$ in this case and extract from it the correlation $C(t, t')$ and the response $R(t, t')$ for an arbitrary t_0 and $x(t_0) = x_0$.