GENERAL RELATIVITY

M2 Theoretical physics

September 19, 2018.

1 2-dimensional sphere

1-a) Consider an ambient space of dimension D with the metric G and some coordinates X^A $(A = 1, 2, 3...D)$. Consider a hypersurface of coordinates x^μ $(\mu = 1, 2..., D-1)$ defined by its equation $X^A(x^\mu)$. Show that G induces, on the hypersurface, a metric

$$g_{\mu\nu} = \frac{\partial X^A}{\partial x^\mu}\frac{\partial X^B}{\partial x^\nu}G_{AB}.$$

Apply that to a two-dimensional sphere of radius R, embedded in the usual Euclidean space.

1-b) Compute the Christoffel symbols, the Ricci tensor and scalar of such a sphere.

1-c) At the point $A = (\theta = \alpha, \varphi = \varphi_0)$, one defines the vector $\vec{T} = (0, 1)$, in the natural basis $(\partial_\theta, \partial_\varphi)$. One considers the parallel transport of $\vec{T}$ to the point $B\,(\theta = \alpha, \varphi = \varphi_0 + \delta)$, along the curve $\theta = \alpha$. Determine the value of $\vec{T}$ at B.

After one round trip, what is the value of $\vec{T}$? Any comments ?

2 Generalities on Killing vector fields

Consider the transformation $x'^\mu = x^\mu + \epsilon\xi^\mu$.

2-a) What change does it induce on the metric ? More precisely, one will compute $g'_{\mu\nu}(x'^\alpha) - g_{\mu\nu}(x'^\alpha)$, at first order in ϵ.

2-b) The metric admits a symmetry, described by ξ^μ, if the quantity found in **2-a)** vanishes. What equation does ξ^μ verify ? Express it in terms of a Lie derivative. If it is fulfilled then ξ^μ is said to be a Killing field.

2-c) Express the Killing equation in terms of the covariant derivative associated to the metric.

2-d) Suppose that, in a given set of coordinates, the metric does not depend on the variable X^n. Show that the metric then admits a Killing vector.

2-e) Let ξ^μ and ω^μ be two Killing fields. Show that any linear combination with constant coefficients is also a Killing vector field.

What of the vector $\Psi^\mu = \xi^\nu\partial_\nu\omega^\mu - \omega^\nu\partial_\nu\xi^\mu$?

2-f) Let $T^{\mu\nu}$ be one symmetric and conserved tensor (i.e. $D_\mu T^{\mu\nu} = 0$). If there exists a Killing vector field, show that one can construct a conserved vector I^μ (i.e. $D_\mu I^\mu = 0$).

One defines $\hat{I}^\mu = \sqrt{-g}I^\mu$. Show that $\partial\mu\hat{I}^\mu = 0$. Deduce that $Q = \int \hat{I}^0 \mathrm{d}^3x$ is conserved, that is such that $\partial_0 Q = 0$, if $\hat{I}$ decreases fast enough at the boundary of the spacetime.

3 Maximally symmetric spaces

3-a) Show that a Killing vector field ξ^μ must obey $D_\mu D_\nu \xi_\rho = R^\sigma_{\mu\nu\rho}\xi_\sigma$.

3-b) Consider a two points A and B joined by a curve. Eq. (3-a) is then a second order ordinary differential equation along that curve. What quantities must be given at A to integrate it ? If one considers all the possible curves from A to B, what are then the possible degrees of freedom ? Deduce the maximum number of independant Killing vectors of a space of dimension n (i.e. that are not constant linear combination of each others).

3-c) Let consider the same sphere as in **1)**. Find its Killing vectors (one will express them in Cartesian coordinates).

3-d) What are the Killing vector fields of Minkowski spacetime ?

One can show that the geometry of maximally symmetric spacetimes depends only on i) the dimension, ii) the signature of the metric and iii) the scalar curvature which is constant. Maximally symmetric spacetimes are the spacetimes of constant scalar curvature.

4 de Sitter spacetime

4-a) Let consider, in the 5-dimensional Minkowski spacetime, the surface defined by

$$-\left(X^0\right)^2+\left(X^1\right)^2+\left(X^2\right)^2+\left(X^3\right)^2+\left(X^4\right)^2=H^{-2}$$

where H is a constant.

Find the metric induced on this hypersurface. One will define

$$\begin{aligned} X^i &= e^{Ht}x^i \quad \text{with} \quad i=1,2,3 \\ X^0-X^4 &= 2e^{Ht} \end{aligned}$$

and write the result as $\mathrm{d}s^2=-\mathrm{d}t^2+a^2\left(t\right)\delta_{ij}\mathrm{d}x_i\mathrm{d}x_j$.

One can use $x^1=x$, $x^2=y$, $x^3=z$ and $r^2=x^2+y^2+z^2$.

This describes the de Sitter spacetime in Lemaitre coordinates. Do they cover the whole hypersurface ?

4-b) Compute the Ricci scalar. Comments ?

4-c) Write Einstein tensor in de Sitter spacetime. Comments ?

4-d) Consider the following set of coordinates

$$\begin{aligned} X^0 &= \frac{1}{H}\sinh\left(H\bar{t}\right)\sqrt{1-H^2\bar{r}^2} \\ X^1 &= x \quad ; \quad X^2=y \quad ; \quad X^3=z \qquad (1) \\ X^4 &= \frac{1}{H}\cosh\left(H\bar{t}\right)\sqrt{1-H^2\bar{r}^2}. \qquad (2) \end{aligned}$$

Show that they do describe de Sitter spacetime. What is the expression the metric ? Comments ?

5 Anti de Sitter spacetime

5-a) Consider the 5-dimensional spacetime with the following metric

$$\mathrm{d}s^2=-\left(\mathrm{d}X^0\right)^2-\left(\mathrm{d}X^4\right)^2+\left(\mathrm{d}X^1\right)^2+\left(\mathrm{d}X^2\right)^2+\left(\mathrm{d}X^3\right)^2.$$

Consider the following set of coordinates :

$$X^0 = \frac{1}{H}\sin\left(Ht\right)\cosh\left(r\right)$$

$$
\begin{aligned}
X^4 &= \frac{1}{H}\cos(Ht)\cosh(r) && (3)\\
X^1 &= \frac{1}{H}\sinh(r)\sin\theta\cos\varphi && (4)\\
X^2 &= \frac{1}{H}\sinh(r)\sin\theta\sin\varphi && (5)\\
X^3 &= \frac{1}{H}\sinh(r)\cos\theta. && (6)
\end{aligned}
$$

What is the equation of the hypersurface described by (t, r, θ, φ). Determine the value of the induced metric.

5-b) Compute the Ricci scalar. Comments ?

5-c) Compute the Einstein tensor. Comments ?

6 Surface gravity

One talks about Killing horizon if a black hole admits a Killing vector field k^μ that is null ($k_\mu k^\mu = 0$) on the horizon. One can then show that, on the horizon, $k^\alpha D_\alpha k^\beta = \kappa k^\beta$, where κ is a constant known as the surface gravity.

Consider the Schwarzschild black hole in Eddington-Finkelstein coordinates:

$$
\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + 2\mathrm{d}r\mathrm{d}t + r^2\mathrm{d}\Omega.
$$

Reminder: the horizon is the sphere located at $r = 2M$.

6-a) Does the Schwarzschild spacetime contain a Killing horizon ? For which vector field ?

6-b) Compute κ.

6-c) Compute κ in the following coordinates (the horizon is also located at $r = 2M$)

$$
\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega.
$$

Comments ?