Solution Set for Exercise Session No.8

Course: Mathematical Aspects of Symmetries in Physics,
ICFP Master Program (for M1)

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Lecture by Amir-Kian Kashani-Poor (email: kashani@lpt.ens.fr)
Exercise Session by Tatsuo Azeyanagi (email: tatsuo.azeyanagi@phys.ens.fr)

1 Integral Curve

Recall that the integral curve of a vector field X on a manifold M is defined as a curve $c:(a,b)\to M$ satisfying $dc_t((d/dt)|_t) = X_{c(t)}$. We denote as $c(t)=(x(t),y(t))$. Then from the definition, by using C^∞ function f, we have

$$\begin{aligned} X_{c(t)}f &= -y(t)\frac{\partial f}{\partial x}\bigg|_{c(t)} + x(t)\frac{\partial f}{\partial y}\bigg|_{c(t)}, \\ dc_t((d/dt)|_t)f &= \frac{d(f\circ c)}{dt}\bigg|_t = \frac{dx}{dt}\bigg|_t\frac{\partial f}{\partial x}\bigg|_{c(t)} + \frac{dy}{dt}\bigg|_t\frac{\partial f}{\partial y}\bigg|_{c(t)}, \end{aligned}$$

and thus by comparing these we obtain

$$\frac{dx}{dt}\bigg|_t = -y(t)\,, \qquad \frac{dy}{dt}\bigg|_t = x(t)\,.$$

By solving this, we obtain (since $d^2x/dt^2 = -x(t)$)

$$x(t) = A\cos t + B\sin t\,, \qquad y(t) = A\sin t - B\cos t\,,$$

where A and B are constants. The initial condition, $(x(0),y(0)) = (x_0,y_0)$ determines A and B as $A = x_0$ and $B = -y_0$. Therefore the integral curve is

$$x(t) = x_0\cos t - y_0\sin t\,, \qquad y(t) = x_0\sin t + y_0\cos t\,.$$

2 Some Property of Exponential Map of Matrix

1. By taking the derivative directly, we obtain

$$\begin{aligned} \frac{d}{dt}e^{tA} &= \frac{d}{dt}\left(1 + tA + \frac{1}{2!}t^2A^2 + \frac{1}{3!}t^3A^3 + \cdots\right) \\ &= A + tA^2 + \frac{1}{2!}t^2A^3 + \cdots \\ &= A\left(1 + tA + \frac{1}{2!}t^2A^2 + \cdots\right) \\ &= Ae^{tA} \end{aligned}$$

$$\begin{aligned} &= \left(1 + tA + \frac{1}{2!}t^2A^2 + \cdots\right) A \\ &= e^{tA}A\,. \end{aligned}$$

2. Since $[A, B] = 0$, we obtain

$$\begin{aligned} e^A e^B &= \sum_{m=0}^{\infty} \frac{1}{m!}(A)^m \sum_{n=0}^{\infty} \frac{1}{n!}(B)^n \\ &= \sum_{l=0}^{\infty}\sum_{k=0}^{l} \frac{1}{k!}A^k \frac{1}{(l-k)!}B^{l-k} \\ &= \sum_{l=0}^{\infty}\frac{1}{l!}\sum_{k=0}^{l} \frac{l!}{k!(l-k)!}A^k B^{l-k} \\ &= \sum_{l=0}^{\infty}\frac{1}{l!}(A+B)^l \\ &= \exp(A+B)\,. \end{aligned}$$

In the middle we have defined $l = m + n$ and $k = m$ and used $[A, B] = 0$

3. We first notice that

$$\begin{aligned} \frac{d^n}{dt^n}\left(e^{tA}Be^{-tA}\right) &= \frac{d^{n-1}}{dt^{n-1}}\left([A, e^{tA}Be^{-tA}]\right) \\ &= \frac{d^{n-2}}{dt^{n-2}}\left([A, [A, e^{tA}Be^{-tA}]]\right) \\ &= \cdots \\ &= [A, \cdots [A, [A, e^{tA}Be^{-tA}]] \cdots]\,. \end{aligned}$$

Here in the final expression there are n $[A, \cdot]$'s. Then by Talyor expanding $e^{tA}Be^{-tA}$ with respect to t around $t = 0$, we obtain

$$e^{tA}Be^{-tA} = B + t[A,B] + \frac{1}{2!}t^2[A,[A,B]] + \frac{1}{3!}t^3[A,[A,[A,B]]] + \cdots\,.$$

Now let us consider the case with $[A, B] = B$. In this case we have $[A, [A, B]] = [A, B] = B$. More generally, we obtain

$$[A, \cdots [A, [A, B]] \cdots] = B\,.$$

Therefore we finally obtain

$$e^{tA}Be^{-tA} = B + tB + \frac{1}{2!}t^2B + \frac{1}{3!}t^3B + \cdots = e^tB\,.$$

4. When $[A, B] = C$ and $[A, C] = B$ are satisfied, we have $[A, [A, B]] = [A, C] = B$ and $[A, [A, [A, B]]] = [A, [A, C]] = [A, B] = C$. Thus when there are even number

of the commutators we have $[A, \cdots [A, [A, B]] \cdots] = B$, while for odd number of them we have $[A, \cdots [A, [A, B]] \cdots] = C$. Therefore, we finally obtain

$$\begin{aligned} e^{tA} B e^{-tA} &= B + tC + \frac{1}{2!}t^2 B + \frac{1}{3!}t^3 C + \cdots \\ &= \left[1 + \frac{1}{2!}t^2 + \cdots\right] B + \left[t + \frac{1}{3!}t^3 + \cdots\right] C \\ &= (\cosh t)\, B + (\sinh t)\, C\,. \end{aligned}$$

5. When A is diagonalizable, we can write A as $A = MDM^{-1}$ where M is an $n \times n$ square matrix and $D = \mathrm{diag}(d_1, d_2, \cdots)$ is a diagonal matrix. Then we have

$$\det(\exp(A)) = \det(MM^{-1}\exp(A)) = \det(M^{-1}\exp(A)M)\,.$$

Since

$$\begin{aligned} M^{-1}\exp(A)M &= M^{-1}\left(\sum_{k=0}^{\infty}\frac{1}{k!}A^k\right)M \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}(M^{-1}AM)^k \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}\begin{pmatrix} d_1^k & 0 & \cdots \\ 0 & d_2^k & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \\ &= \begin{pmatrix} \exp(d_1) & 0 & \cdots \\ 0 & \exp(d_2) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \end{aligned}$$

In the second equality, we have inserted $MM^{-1} = \mathbf{1}_n$ between A and A (here $\mathbf{1}_n$ is the$n \times n$ unit matrix). Thus we have $\det(\exp(A)) = \prod_i \exp(d_i) = \exp(\sum_i d_i)$.

On the other hand, we have

$$\exp(\mathrm{tr}A) = \exp(\mathrm{tr}(MM^{-1}A)) = \exp(\mathrm{tr}(M^{-1}AM)) = \exp\left(\sum_i d_i\right).$$

Therefore we have proved the desired relation.

6. In general, one can take an appropriate $n \times n$ matrix M to write A as

$$A = MJM^{-1}\,,$$

where J is the Jordan canonical form

$$J = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{pmatrix}, \qquad \text{with} \qquad J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix},$$

where λ_i is a number and J_i is a $n_i \times n_i$ matrix ($i = 1, 2, \cdots k$). We note that $n = \sum_{i=1}^{k} n_i$. We note that J_i is written as $J_i = \lambda_i \mathbf{1}_{n_i} + N_i$ where $\mathbf{1}_{n_i}$ is the $n_i \times n_i$ unit matrix and

$$N_i = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} .$$

This N_i satisfies $(N_i)^{n_i} = 0$ and obviously $[\mathbf{1}_{n_i}, N_i] = 0$. We also note that $\mathrm{tr} N_i = 0$

Now we can compute $\det(\exp(A))$ as

$$\begin{aligned} \det(\exp(A)) &= \det(\exp(M^{-1}AM)) \\ &= \prod_{i=1}^{k} \det(\exp(\lambda_i \mathbf{1}_{n_i} + N_i)) \\ &= \prod_{i=1}^{k} \det(\exp(\lambda_i \mathbf{1}_{n_i}) \exp(N_i)) \\ &= \prod_{i=1}^{k} \det(\exp(\lambda_i \mathbf{1}_{n_i})) \det(\exp(N_i)) \\ &= \prod_{i=1}^{k} \exp(n_i \lambda_i) \det(\exp(N_i)) . \end{aligned}$$

Now we evaluate $\det(\exp(N_i))$. Since

$$\begin{aligned} \exp(N_i) &= \sum_{m=0}^{\infty} \frac{1}{m!} (N_i)^m \\ &= \sum_{m=0}^{n_i - 1} \frac{1}{m!} (N_i)^m \\ &= \begin{pmatrix} 1 & * & \cdots & \cdots & * \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & 1 & \ddots & \vdots \\ \vdots & & \ddots & 1 & * \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} . \end{aligned}$$

Because of this upper triangle form, we can easily obtain $\det(\exp(N_i)) = 1$. Therefore, we have $\det(\exp(A)) = \prod_{i=1}^{k} \exp(n_i \lambda_i)$.

On the other hand, we can compute $\exp(\mathrm{tr} A)$ as

$$\exp(\mathrm{tr} A) = \exp(\mathrm{tr} M^{-1} AM)$$

$$
\begin{aligned}
&= \exp\left(\sum_{j=1}^{k} \mathrm{tr}(\lambda_i \mathbf{1}_{n_i} + N_i)\right) \\
&= \exp\left(\sum_{i=1}^{k} \mathrm{tr}(\lambda_i \mathbf{1}_{n_i})\right) \\
&= \prod_{i=1}^{k} \exp\left(\mathrm{tr}(\lambda_i \mathbf{1}_{n_i})\right) \\
&= \prod_{i=1}^{k} \exp\left(n_i \lambda_i\right) .
\end{aligned}
$$

Therefore, we have proved the desired relation.

3 Lie Group and Lie Algebra

(1) We first denote the left-invariant vector field corresponding to X as $\tilde{X}$, and $\tilde{X}$ at $g \in GL(n, \mathbb{R})$ is denoted as $\tilde{X}_g$. As we have seen in Problem Set No.7, we have $\tilde{X}_g = gX$ where gX is defined as $\sum_{i,j,k} c_{ik}(t) A_{kj} \frac{\partial f}{\partial x_{ij}}\Big|_{c(t)}$. Now we derive the integral curve $c(t) : (a, b) \to GL(n, \mathbb{R})$ by definition satisfying (for a C^∞ function f on $\mathrm{GL}(n, \mathbb{R})$)

$$
dc_t \left(\frac{d}{dt}\bigg|_t \right) f = \tilde{X}_{c(t)} f = \sum_{i,j,k} c_{ik}(t) A_{kj} \frac{\partial f}{\partial x_{ij}}\bigg|_{c(t)} .
$$

We also note that

$$
dc_t \left(\frac{d}{dt}\bigg|_t \right) f = \frac{d(f \circ c)}{dt}\bigg|_t = \sum_{i,j} \frac{dc_{ij}}{dt}\bigg|_t \frac{\partial f}{\partial x_{ij}}\bigg|_{c(t)} .
$$

Thus we obtain

$$
\frac{dc}{dt}\bigg|_t = c(t) A \,.
$$

The solution of this equation satisfying $c(t) = I_n$ is

$$
c(t) = \exp(tA) = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n \,.
$$

(2)

1. We notice that $(c(t))^T c(t) = I_n$ from which we obtain (by taking the derivative with respect to t and denoting $(d/dt)c$ evaluated at t as $c'(t)$)

$$
(c'(t))^T c(t) + c(t) c'(t) = 0 \,.
$$

By by using $c(t) = \exp(tA)$ and $(c(t))^T = \exp(tA^T)$ and evaluating this at $t = 0$, we obtain

$$A^T + A = 0\,.$$

Thus A is $n \times n$ real matrix satisfying $A_{ij} = 0$ for $i = j$ and $A_{ij} = -A_{ji}$ for $i \neq j$

NOTE: I think I happened to skip the following explanation on $\det c(t) = 1$ condition in the exercise session. Sorry...

We also note that $\det c(t) = 1$. Since $\det(\exp(M)) = \exp(\mathrm{tr}M)$ for a general square matrix M, we have from $\det c(t) = 1$

$$\exp(\mathrm{tr}(tA)) = 1\,.$$

Since $\mathrm{tr}(tA) = t\sum_i A_{ii} = 0$, this equality is satisfied automatically for X satisfying $A^T + A = 0$.

2. From the previous problem, we can see that A is an $n \times n$ real matrix satisfying $A_{ij} = 0$ for $i = j$ and $A_{ij} = -A_{ji}$ for $i \neq j$. Thus, there are $n(n-1)/2$ independent components in A. We thus conclude that $\dim \mathfrak{so}(n, \mathbb{R}) = \dim T_{I_n} SO(n, \mathbb{R}) = n(n-1)/2$.

3. From the above problem, it is obvious that one can write A as given in the problem. One can also evaluate the commutators straightforwardly.

(3) In a similar way, we use the result from (1). We notice that $(c(t))^T J c(t) = J$ where $c(t) = \exp(tA)$. Then by taking the derivative with respect to t and setting $t = 0$, we obtain

$$A^T J + JA = 0\,.$$

Now we denote A as

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix},$$

where p, q, r, s are real $n \times n$ matrices. Then the above relation for A is equivalent to

$$\begin{aligned} &\begin{pmatrix} p^T & q^T \\ r^T & s^T \end{pmatrix} \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} + \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = 0 \\ \leftrightarrow \quad &\begin{pmatrix} -r^T & p^T \\ -s^T & q^T \end{pmatrix} = -\begin{pmatrix} r & s \\ -p & -q \end{pmatrix}. \end{aligned}$$

We first note that q satisfies $q^T = q$ and thus q has $n(n-1)/2 + n = n(n+1)/2$ independent components. The r also satisfies $r^T = r$ and thus has $n(n-1)/2 + n = n(n+1)/2$ independent components. On the other hand, p, s satisfy $p^T = -s$. Thus s is determined completed once p is determined. Thus in p and s, there are n^2 independent components in total. To summarize, $\dim \mathfrak{sp}(2n, \mathbb{R}) = \dim T_{I_{2n}} Sp(2n, \mathbb{R}) = n(n+1)/2 \times 2 + n^2 = 2n^2 + n$