Solution Set for Exercise Session No.7

Course: Mathematical Aspects of Symmetries in Physics,
ICFP Master Program (for M1)

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1 Lie Bracket

(1) We have at each point in $\mathbb{R}^3$, for C^∞ function f, (in the following we evaluate at each point $p \in \mathbb{R}^3$, but we do not write it explicitly to simplify the notation)

$$\begin{aligned}
[X,Y]f &= X(Yf) - Y(Xf) \\
&= (x\partial_x - y\partial_y + z\partial_z)(x\partial_y f) - x\partial_y(x\partial_x f - y\partial_y f + z\partial_z f) \\
&= x\partial_y f + x(x\partial_x - y\partial_y + z\partial_z)(\partial_y f) + x\partial_y f - x(x\partial_x - y\partial_y + z\partial_z)(\partial_y f) \\
&= 2x\partial_y f \\
&= 2Yf\,.
\end{aligned}$$

Thus we have obtained $[X,Y] = 2Y$. In a similar way, we have

$$\begin{aligned}
[Y,Z]f &= Y(Zf) - Z(Yf) \\
&= (x\partial_y)\left(y\partial_x f + \frac{1+yz}{x}\partial_z f\right) - \left(y\partial_x + \frac{1+yz}{x}\partial_z\right)(x\partial_y f) \\
&= x\partial_x f + z\partial_z f - y\partial_y f \\
&= Xf\,,
\end{aligned}$$

$$\begin{aligned}
[Z,X]f &= Z(Xf) - X(Zf) \\
&= \left(y\partial_x + \frac{1+yz}{x}\partial_z\right)(x\partial_x f - y\partial_y f + z\partial_z f) - (x\partial_x - y\partial_y + z\partial_z)\left(y\partial_x f + \frac{1+yz}{x}\partial_z f\right) \\
&= y\partial_x f + \frac{1+yz}{x}\partial_z f - \left(-x\frac{1+yz}{x^2}\partial_z f - y\partial_x f - y\frac{z}{x}\partial_z f + z\frac{y}{x}\partial_z f\right) \\
&= 2y\partial_x f + 2\frac{1+yz}{x}\partial_z f \\
&= 2Zf\,,
\end{aligned}$$

and thus we have obtained $[Y,Z] = X$ and $[Z,X] = 2Z$.
(2) Following the lecture, we take a local coordinate in the neighborhood of a point $p \in M$ and denote it as x^i. By using this we write the vector fields X, Y, Z on the neighborhood can be denoted as

$$X = \sum_i \xi^i \frac{\partial}{\partial x^i}\,, \qquad Y = \sum_i \eta^j \frac{\partial}{\partial x^i}\,, \qquad Z = \sum_i \chi^j \frac{\partial}{\partial x^i}\,.$$

We also take an C^∞ function h in this neighborhood of p. Then we can evaluate (in the following we evaluate at point p, but do not write it explicitly to simplify the notation)

$$Yh = \sum_j \eta^j \frac{\partial h}{\partial x^j} ,$$

and then

$$X(Yh) = \sum_{i,j} \xi^i \frac{\partial}{\partial x^i} \left(\eta^j \frac{\partial h}{\partial x^j} \right) = \sum_{i,j} \left(\xi^i \eta^j \frac{\partial^2 h}{\partial x^i \partial x^j} + \xi^i \frac{\partial \eta^j}{\partial x^i} \frac{\partial h}{\partial x^j} \right) .$$

Then $[X,Y]h$ becomes

$$[X,Y]h = \sum_{i,j} \left((\xi^i \eta^j - \eta^i \xi^j) \frac{\partial^2 h}{\partial x^i \partial x^j} + \left(\xi^i \frac{\partial \eta^j}{\partial x^i} - \eta^i \frac{\partial \xi^j}{\partial x^i} \right) \frac{\partial h}{\partial x_j} \right) = \sum_{i,j} \left(\xi^i \frac{\partial \eta^j}{\partial x^i} - \eta^i \frac{\partial \xi^j}{\partial x^i} \right) \frac{\partial h}{\partial x_j} .$$

Thus

$$[X,Y] = \sum_j \sum_i \left(\xi^i \frac{\partial \eta^j}{\partial x^i} - \eta^i \frac{\partial \xi^j}{\partial x^i} \right) \frac{\partial}{\partial x_j} .$$

The first three identities can be checked easily from the above expression of the Lie bracket. We will just consider the 4th and 5th identities below.

We first prove the 4th identity. We can have $[Z,[X,Y]]$ as

$$\begin{aligned}
[Z,[X,Y]] &= \sum_{i,j} \left(\chi^i \frac{\partial}{\partial x^i} \left(\sum_k \left(\xi^k \frac{\partial \eta^j}{\partial x^k} - \eta^k \frac{\partial \xi^j}{\partial x^k} \right) \right) - \sum_k \left(\xi^k \frac{\partial \eta^i}{\partial x^k} - \eta^k \frac{\partial \xi^i}{\partial x^k} \right) \frac{\partial \chi^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} \\
&= \sum_{i,j,k} \left(\chi^i \xi^k \frac{\partial^2 \eta^j}{\partial x^i \partial x^k} - \chi^i \eta^k \frac{\partial^2 \xi^j}{\partial x^i \partial x^k} \right) \frac{\partial}{\partial x^j} \\
&\quad + \sum_{i,j,k} \left(\chi^i \frac{\partial \xi^k}{\partial x^i} \frac{\partial \eta^j}{\partial x^k} - \chi^i \frac{\partial \eta^k}{\partial x^i} \frac{\partial \xi^j}{\partial x^k} - \xi^k \frac{\partial \eta^i}{\partial x^k} \frac{\partial \chi^j}{\partial x^i} + \eta^k \frac{\partial \xi^i}{\partial x^k} \frac{\partial \chi^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} .
\end{aligned}$$

In a similar way we obtain

$$\begin{aligned}
[X,[Y,Z]] &= \sum_{i,j,k} \left(\xi^i \eta^k \frac{\partial^2 \chi^j}{\partial x^i \partial x^k} - \xi^i \chi^k \frac{\partial^2 \eta^j}{\partial x^i \partial x^k} \right) \frac{\partial}{\partial x^j} \\
&\quad + \sum_{i,j,k} \left(\xi^i \frac{\partial \eta^k}{\partial x^i} \frac{\partial \chi^j}{\partial x^k} - \xi^i \frac{\partial \chi^k}{\partial x^i} \frac{\partial \eta^j}{\partial x^k} - \eta^k \frac{\partial \chi^i}{\partial x^k} \frac{\partial \xi^j}{\partial x^i} + \chi^k \frac{\partial \eta^i}{\partial x^k} \frac{\partial \xi^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} ,
\end{aligned}$$

$$\begin{aligned}
[Y,[Z,X]] &= \sum_{i,j,k} \left(\eta^i \chi^k \frac{\partial^2 \xi^j}{\partial x^i \partial x^k} - \eta^i \xi^k \frac{\partial^2 \chi^j}{\partial x^i \partial x^k} \right) \frac{\partial}{\partial x^j} \\
&\quad + \sum_{i,j,k} \left(\eta^i \frac{\partial \chi^k}{\partial x^i} \frac{\partial \xi^j}{\partial x^k} - \eta^i \frac{\partial \xi^k}{\partial x^i} \frac{\partial \chi^j}{\partial x^k} - \chi^k \frac{\partial \xi^i}{\partial x^k} \frac{\partial \eta^j}{\partial x^i} + \xi^k \frac{\partial \chi^i}{\partial x^k} \frac{\partial \eta^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} .
\end{aligned}$$

By summing them up, we can see that $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$.

Now we go to the proof of the 5th identity. We first note that

$$(gY)h = g\sum_j \eta^j \frac{\partial h}{\partial x^j}\,,$$

and then

$$\begin{aligned}
(fX)((gY)h) &= f\sum_i \xi^i \frac{\partial}{\partial x^i}\left(g\sum_j \eta^j \frac{\partial h}{\partial x^j}\right) \\
&= f\sum_{i,j}\left(\xi^i \frac{\partial g}{\partial x^i}\eta^j \frac{\partial h}{\partial x^j} + \xi^i g \frac{\partial \eta^j}{\partial x^i}\frac{\partial h}{\partial x^j} + g\eta^i\xi^j \frac{\partial^2 h}{\partial x^i \partial x^j}\right).
\end{aligned}$$

In a similar way, we have

$$(gY)((fX)h) = g\sum_{i,j}\left(\eta^i \frac{\partial f}{\partial x^i}\xi^j \frac{\partial h}{\partial x^j} + \eta^i f \frac{\partial \xi^j}{\partial x^i}\frac{\partial h}{\partial x^j} + f\xi^i\eta^j \frac{\partial^2 h}{\partial x^i \partial x^j}\right).$$

Then we have

$$\begin{aligned}
[fX,gY]h &= fg\sum_{i,j}\left(\xi^i \frac{\partial \eta^j}{\partial x^i} - \eta^i \frac{\partial \xi^j}{\partial x^i}\right)\frac{\partial h}{\partial x^j} + f\sum_{i,j}\left(\xi^i \frac{\partial g}{\partial x^i}\right)\eta^j \frac{\partial h}{\partial x^j} - g\sum_{i,j}\left(\eta^i \frac{\partial f}{\partial x^i}\right)\xi^j \frac{\partial h}{\partial x^j} \\
&= fg[X,Y]h + f(Xg)Yh - g(Yf)Xh\,.
\end{aligned}$$

Thus we have confirmed that $[fX,gY]=fg[X,Y]+f(Xg)Y-g(Yf)X$.

2 f-related Vector Field

1. We first compute the Jacobian of f at $p=(x,y)\in\mathbb{R}^2$ (we note that $f(p)=(x,x^2+y)\equiv(X,Y)$):

$$Df|p = \begin{pmatrix} 1 & 0 \\ 2x & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2X & 1 \end{pmatrix}.$$

Thus, f-related vector field of V is obtained as follows: at each point $p=(x,y)\in\mathbb{R}^2$ for a C^∞ function g, we have (here f_1, f_2 are defined such that $f_1(x), f_2(x)$ are 1st and 2nd components of $f(x)$, respectively)

$$\begin{aligned}
df_p(V|_p)\,g &= V(g\circ f)|_p \\
&= (-y)\left[\left.\frac{\partial f_1}{\partial x}\right|_p \left.\frac{\partial g}{\partial X}\right|_{f(p)} + \left.\frac{\partial f_2}{\partial x}\right|_p \left.\frac{\partial g}{\partial Y}\right|_{f(p)}\right] + x\left[\left.\frac{\partial f_1}{\partial y}\right|_p \left.\frac{\partial g}{\partial X}\right|_{f(p)} + \left.\frac{\partial f_2}{\partial y}\right|_p \left.\frac{\partial g}{\partial Y}\right|_{f(p)}\right] \\
&= -(Y-X^2)\left[1\times\left.\frac{\partial g}{\partial X}\right|_{f(p)} + 2X\times\left.\frac{\partial g}{\partial Y}\right|_{f(p)}\right] + X\left[0\times\left.\frac{\partial g}{\partial X}\right|_{f(p)} + 1\times\left.\frac{\partial g}{\partial Y}\right|_{f(p)}\right]
\end{aligned}$$

$$
\begin{aligned}
&= -(Y-X^2)\frac{\partial g}{\partial X} + X\left(1-2(Y-X^2)\right)\frac{\partial g}{\partial Y}\bigg|_{f(p)} \\
&= \left[-(Y-X^2)\frac{\partial}{\partial X} + X\left(1-2(Y-X^2)\right)\frac{\partial}{\partial Y}\right] g\bigg|_{f(p)} .
\end{aligned}
$$

Thus we have obtained

$$df(V) = -(Y-X^2)\frac{\partial}{\partial X} + X\left(1-2(Y-X^2)\right)\frac{\partial}{\partial Y} .$$

2. Let us first recall that the stereographic protection from the north pole $(0,0,1)$ and south pole $(0,0,-1)$ and their inverses are given by $(U^\pm = S^2 \backslash \{(0,0,\pm 1)\})$

$$
\begin{aligned}
f^+(x,y,z) &= \left(\frac{x}{1-z}, \frac{y}{1-z}\right), \qquad \text{for} \quad (x,y,z) \in U^+ , \\
f^-(x,y,z) &= \left(\frac{x}{1+z}, \frac{y}{1+z}\right), \qquad \text{for} \quad (x,y,z) \in U^- , \\
(f^+)^{-1}(X,Y) &= \left(\frac{2X}{X^2+Y^2+1}, \frac{2Y}{X^2+Y^2+1}, \frac{X^2+Y^2-1}{X^2+Y^2+1}\right), \\
(f^-)^{-1}(X',Y') &= \left(\frac{2X'}{X'^2+Y'^2+1}, \frac{2Y'}{X'^2+Y'^2+1}, \frac{1-X'^2-Y'^2}{X'^2+Y'^2+1}\right),
\end{aligned}
$$

for $(X,Y) \in f^+(U^+) = \mathbb{R}^2$ and $(X',Y') \in f^-(U^-) = \mathbb{R}^2$. The transition function from $f^+(U^+ \cap U^-)$ to $f^-(U^+ \cap U^-)$ is given by

$$\phi_{-+}(X,Y) \equiv f^- \circ (f^+)^{-1}(X,Y) = \left(\frac{X}{X^2+Y^2}, \frac{Y}{X^2+Y^2}\right),$$

for $(X,Y) \in f^+(U^+) = \mathbb{R}^2$. For later use, we also write down the Jacobian matrix for $(f^+)^{-1}$ at $f^+(p) = (x/(1-z), y/(1-z)) \equiv (X,Y) \in f^+(U^+)$ where $p = (x,y,z) \in U^+$:

$$D(f^+)^{-1}|_{f^+(p)} = \begin{pmatrix} \frac{2(1-X^2+Y^2)}{(1+X^2+Y^2)^2} & \frac{-4XY}{(1+X^2+Y^2)^2} \\ \frac{-4XY}{(1+X^2+Y^2)^2} & \frac{2(1+X^2-Y^2)}{(1+X^2+Y^2)^2} \\ \frac{-4X}{(1+X^2+Y^2)^2} & \frac{-4Y}{(1+X^2+Y^2)^2} \end{pmatrix} = \begin{pmatrix} 1-z-x^2 & -xy \\ -xy & 1-z-y^2 \\ -x(1-z) & -y(1-z) \end{pmatrix} .$$

(We note that $x^2+y^2+z^2=1$.) We also write down the Jacobian matrix for ϕ_{-+} at $(X,Y) \in f^+(U^+ \cap U^-)$:

$$D(f^- \circ (f^+)^{-1})|_{(X,Y)} = \begin{pmatrix} \frac{X^2-Y^2}{(X^2+Y^2)^2} & -\frac{2XY}{(X^2+Y^2)^2} \\ -\frac{2XY}{(X^2+Y^2)^2} & \frac{-X^2+Y^2}{(X^2+Y^2)^2} \end{pmatrix} .$$

Now we construct a vector field that vanishes only at the north pole of S^2. We first take a C^∞ vector field $V = \partial_X$ on $f^+(U^+)$. We can then construct the $(f^+)^{-1}$-related vector field of V (this vector field is defined on U^+). For a C^∞ function g on U^+, we have (we abbreviated $|_p$ etc. to simplify the notation)

$$d((f^+)^{-1})(V)g$$

$$
\begin{aligned}
&= V(g \circ (f^+)^{-1}) \\
&= \left(\frac{\partial((f^+)^{-1})_1}{\partial X}\right)\left(\frac{\partial g}{\partial x}\right) + \left(\frac{\partial((f^+)^{-1})_2}{\partial X}\right)\left(\frac{\partial g}{\partial y}\right) + \left(\frac{\partial((f^+)^{-1})_3}{\partial X}\right)\left(\frac{\partial g}{\partial z}\right) \\
&= \left[\frac{2(1-X^2+Y^2)}{(1+X^2+Y^2)^2}\partial_x + \frac{-4XY}{(1+X^2+Y^2)^2}\partial_y + \frac{-4X}{(1+X^2+Y^2)^2}\partial_z\right] g \\
&= \left[(1-z-x^2)\partial_x - xy\partial_y - x(1-z)\partial_z\right] g\,.
\end{aligned}
$$

(Here $((f^+)^{-1})_i$ $(i = 1,2,3)$ is defined such that $((f^+)^{-1})_i(X,Y)$ are the i-th component of $(f^+)^{-1}(X,Y)$.) Here we have used the Jacobian obtained above. From this expression, $d((f^+)^{-1})(V)$ is smooth on U^+ and non-vanishing at any point on U^+.

We next confirm that this vector field is smoothly extended to the north pole $(0,0,1)$ and the extension vanishes at the north pole. To see this, we consider ϕ_{-+}-related vector field of V (this vector field is defined on $f^-(U^+ \cap U^-) = \mathbb{R}^2 \backslash \{(0,0)\}$). For an arbitrary C^∞ function h on $f^-(U^+ \cap U^-)$, we have (here we abbreviated $|_p$ etc. to simplify the notation)

$$
\begin{aligned}
d(\phi_{-+})(V)h &= V(h \circ \phi_{-+}) \\
&= \left(\frac{(\phi_{-+})_1}{\partial X}\right)\left(\frac{\partial h}{\partial X'}\right) + \left(\frac{(\phi_{-+})_2}{\partial X}\right)\left(\frac{\partial h}{\partial Y'}\right) \\
&= \left[\frac{X^2-Y^2}{(X^2+Y^2)^2}\partial_{X'} + \frac{-2XY}{(X^2+Y^2)^2}\partial_{Y'}\right] h \\
&= \left[((X')^2-(Y')^2)\partial_{X'} - 2X'Y'\partial_{Y'}\right] h\,.
\end{aligned}
$$

(Here $(\phi_{-+})_1, (\phi_{-+})_2$ are defined such that $(\phi_{-+})_1(X,Y), (\phi_{-+})_2(X,Y)$ are 1st and 2nd components of $\phi_{-+}(X,Y)$, respectively.) From this expression, we can smoothly extend $d(\phi_{-+})(V)$ to $(X',Y') = (0,0)$ which corresponds to the north pole of S^2. Thus the vector field $d((f^+)^{-1})(V)$ is also smoothly extended to S^2. In particular, since the extension of $d(\phi_{-+})(V)$ vanishes at $(X',Y') = (0,0)$, then the extension of $d((f^+)^{-1})(V)$ vanishes at the north pole. To summarize, the vector W defined by

$$
W = \begin{cases} 0 & \text{at north pole}\,, \\ d((f^+)^{-1})(V) & \text{otherwise}\,, \end{cases}
$$

is a C^∞ vector field on S^2 which vanishes only at the north pole.

——memo—-

Hairy Ball Theorem

On even-dimensional spheres, there is no non vanishing continuous tangent vector field.

3 Left-invariant Vector Field

1. We first recall that for $X \in T_{I_n}GL(n,\mathbb{R})$ there exists a curve $c : (a,b) \to GL(n,\mathbb{R})$ satisfying $c(t=0) = I_n$ and

$$Xf = \left.\frac{d(f \circ c)}{dt}\right|_{t=0} = (dc)_{t=0}\left(\left.\frac{d}{dt}\right|_{t=0}\right) f\,,$$

for an arbitrary C^∞ function f in the neighborhood of I_n, and thus

$$X = (dc)_{t=0}\left(\left.\frac{d}{dt}\right|_{t=0}\right).$$

We also notice that

$$\left.\frac{d(f \circ c)}{dt}\right|_{t=0} = \sum_{i,j}\left.\frac{dc_{ij}}{dt}\right|_{t=0}\left.\frac{\partial f}{\partial x_{ij}}\right|_{I_n},$$

(Here we denoted (i,j)-component of $c(t)$ as $c_{ij}(t)$ for simplicity. In other words, by denoting the projection of $g \in GL(n,\mathbb{R})$ to (i,j)-component as x_{ij}, $c_{ij} = x_{ij} \circ c$) and thus

$$A_{ij} = \left.\frac{dc_{ij}}{dt}\right|_{t=0}.$$

Now we prove the relation in the problem. For C^∞ function h in the neighborhood of g, we have

$$\begin{aligned}
(d\ell_g)_{I_n}(X)h &= (d\ell_g)_{I_n}\left((dc)_{t=0}\left(\left.\frac{d}{dt}\right|_{t=0}\right)\right) h \\
&= (d(\ell_g \circ c))_{t=0}\left(\left.\frac{d}{dt}\right|_{t=0}\right) h \\
&= \left.\frac{d(h \circ \ell_g \circ c)}{dt}\right|_{t=0} \\
&= \sum_{i,j}\left.\frac{d(\ell_g \circ c)_{ij}}{dt}\right|_{t=0}\left.\frac{\partial h}{\partial x_{ij}}\right|_{\ell_g \circ c(t=0)=g} \\
&= \sum_{i,j,k} g_{ik}\left.\frac{dc_{kj}}{dt}\right|_{t=0}\left.\frac{\partial h}{\partial x_{ij}}\right|_{g} \\
&= \sum_{i,j,k} g_{ik}A_{kj}\left.\frac{\partial h}{\partial x_{ij}}\right|_{g} \\
&= (gX)h\,.
\end{aligned}$$

Here gX is what we have defined in the problem. Thus we conclude that $(d\ell_g)_{I_n}(X) = gX$.

2. Now we notice that, by the definition of the left-invariant vector field, we have (we denote $\tilde{X}$ evaluated at $g \in GL(n, \mathbb{R})$ as $\tilde{X}_g$)

$$(d\ell_g)_x(\tilde{X}_x) = \tilde{X}_{gx} \,,$$

for $x, g \in GL(n, \mathbb{R})$. By setting $x = I_n$, we obtain

$$(d\ell_g)_{I_n}(X) = \tilde{X}_{gI_n} = \tilde{X}_g \,.$$

Here we have used $\tilde{X}_{g=I_n} = X$. Comparing with the result of the previous problem, we conclude that $\tilde{X}_g = gX$.