Solution Set for Exercise Session No.4

Course: Mathematical Aspects of Symmetries in Physics,
ICFP Master Program (for M1)
11th December, 2014, at Room 235A
Lecture by Amir-Kian Kashani-Poor (email: kashani@lpt.ens.fr)
Exercise Session by Tatsuo Azeyanagi (email: tatsuo.azeyanagi@phys.ens.fr)

1 Character for Representation of D_3

(1)Let us now take the trace of $M(g)$:

$$\chi(g) = \mathrm{tr}(M(g)) = \sum_a m_a \mathrm{tr}(M^{(a)}(g)) = \sum_a m_a \chi^{(a)}(g)\,.$$

By multiplying $\overline{\chi^{(b)}(g)}$ and then taking the sum over $g \in G$, we have

$$\sum_{g\in G}\overline{\chi^{(b)}(g)}\chi(g) = \sum_a m_a \sum_{g\in G}\overline{\chi^{(b)}(g)}\chi^{(a)}(g) = r\sum_a \delta_{ab} m_a = r\, m_b\,.$$

Here we have used the orthogonality relation for $\chi^{(a)}(g)$:

$$\frac{1}{r}\sum_{g\in G}\overline{\chi^{(b)}(g)}\chi^{(a)}(g) = \delta_{ab}\,.$$

On the other hand, we recall that each element of G belongs to one and only one of n_c conjugacy classes (Let us assume that $g \in G$ belongs to two inequivalent conjugacy classes C_i and C_j with representatives g_i and g_j, respectively ($C_i \neq C_j$). Then we have $g = a g_i a^{-1}$ and $g = b g_j b^{-1}$ for some $a, b \in G$. This leads to $g_i = (a^{-1}b) g_j (a^{-1}b)^{-1}$. From this we can see that $\forall h \in C_i$ belongs to C_j as well. Thus we have $C_i \subset C_j$. We can also show $C_i \supset C_j$. Thus we conclude that $C_i = C_j$. Therefore, each element of G belongs to one and only one of n_c conjugacy classes). Thus we can rewrite the sum $\sum_{g\in G}$ as $\sum_{i=1}^{n_c}\sum_{g\in C_i}$. By recalling that the character takes the same value for the elements in the same class, we have

$$\sum_{g\in G}\overline{\chi^{(b)}(g)}\chi(g) = \sum_{i=1}^{n_c}\sum_{g\in C_i}\overline{\chi^{(b)}(g)}\chi(g) = \sum_{i=1}^{n_c} r_i\,\overline{\chi^{(b)}(C_i)}\chi(C_i)\,.$$

Therefore, we conclude that

$$r\, m_b = \sum_{i=1}^{n_c} r_i\,\overline{\chi^{(b)}(C_i)}\chi(C_i)\,.$$

By replacing b by a and dividing both sides by r, we obtain the desired expression.
(2)

1. Taking the trace of the representation matrices, we obtain

$$\chi(e) = 1+1+1 = 3\,, \qquad \chi(c_3) = 0+0+0 = 0\,, \qquad \chi((c_3)^{-1}) = 0+0+0 = 0\,,$$
$$\chi(\sigma_1) = 1+0+0 = 1\,, \qquad \chi(\sigma_2) = 0+1+0 = 1\,, \qquad \chi(\sigma_3) = 0+0+1 = 1\,.$$

(It is enough to compute for a representative of each class, but here we explicitly computed everything.) That is, we have obtained

$$\chi(C_1) = 3\,, \qquad \chi(C_2) = 0\,, \qquad \chi(C_3) = 1\,.$$

2. By taking the trace of the matrix representations $R^{(1)}$, $R^{(1')}$ and $R^{(2)}$ for the representation $\rho^{(1)}$, $\rho^{(1')}$ and $\rho^{(2)}$ as in the previous problem, we obtain

$$\chi^{(1)}(e) = \chi^{(1)}(c_3) = \chi^{(1)}((c_3)^{-1}) = \chi^{(1)}(\sigma_1) = \chi^{(1)}(\sigma_2) = \chi^{(1)}(\sigma_3) = 1\,,$$
$$\chi^{(1')}(e) = \chi^{(1')}(c_3) = \chi^{(1')}((c_3)^{-1}) = 1\,, \quad \chi^{(1')}(\sigma_1) = \chi^{(1')}(\sigma_2) = \chi^{(1')}(\sigma_3) = -1\,,$$
$$\chi^{(2)}(e) = 1+1 = 2\,, \quad \chi^{(2)}(c_3) = -1/2-1/2 = -1\,, \quad \chi^{(2)}((c_3)^{-1}) = -1/2-1/2 = -1\,,$$
$$\chi^{(2)}(\sigma_1) = 1-1 = 0\,, \quad \chi^{(2)}(\sigma_2) = -1/2+1/2 = 0\,, \quad \chi^{(2)}(\sigma_3) = -1/2+1/2 = 0\,.$$

Thus we obtain

$$\chi^{(1)}(C_1) = \chi^{(1)}(C_2) = \chi^{(1)}(C_3) = 1\,,$$
$$\chi^{(1')}(C_1) = \chi^{(1')}(C_2) = 1\,, \qquad \chi^{(1')}(C_3) = -1\,,$$
$$\chi^{(2)}(C_1) = 2\,, \qquad \chi^{(2)}(C_2) = -1\,, \qquad \chi^{(2)}(C_3) = 0\,.$$

Since $r_1 = 1$, $r_2 = 2$ and $r_3 = 3$, the orthogonality relation is

$$\sum_{i=1}^{3} r_i \overline{\chi^{(a)}(C_i)} \chi^{(b)}(C_i) = r\,\delta_{ab}$$
$$\leftrightarrow \quad \overline{\chi^{(a)}(C_1)}\chi^{(b)}(C_1) + 2\overline{\chi^{(a)}(C_2)}\chi^{(b)}(C_2) + 3\overline{\chi^{(a)}(C_3)}\chi^{(b)}(C_3) = 6\,\delta_{ab}\,.$$

For each pair among $R^{(1)}, R^{(1')}$ and $R^{(2)}$, we can compute this left hand side as

$$\begin{aligned}
(a,b) \quad &: \quad \overline{\chi^{(a)}(C_1)}\chi^{(b)}(C_1) + 2\overline{\chi^{(a)}(C_2)}\chi^{(b)}(C_2) + 3\overline{\chi^{(a)}(C_3)}\chi^{(b)}(C_3)\\
(1,1) \quad &: \quad 1\times 1 + 2(1\times 1) + 3(1\times 1) = 6\,,\\
(1',1') \quad &: \quad 1\times 1 + 2(1\times 1) + 3((-1)\times(-1)) = 6\,,\\
(2,2) \quad &: \quad 2\times 2 + 2((-1)\times(-1)) + 3(0\times 0) = 6\,,\\
(1,1') \quad &: \quad 1\times 1 + 2(1\times 1) + 3(1\times(-1)) = 0\,,\\
(1,2) \quad &: \quad 1\times 2 + 2(1\times(-1)) + 3(1\times 0) = 0\,,\\
(1',2) \quad &: \quad 1\times 2 + 2(1\times(-1)) + 3((-1)\times 0) = 0\,.
\end{aligned}$$

Therefore, we have explicitly confirmed the orthogonality of the characters for the irreducible representations.

3. The simplest way to show is just to use the fact that the number of the irreducible representation is the same as the number of the conjugacy classes, as proved in the lecture.

   Here is another way to confirm this. We denote the dimensions of the representations $\rho^{(1)}$, $\rho^{(1')}$ and $\rho^{(2)}$ as $d_1, d_{1'}$ and d_2, respectively. Since $d_1 = d_{1'} = 1$ and $d_2 = 2$, we obtain

$$d_1^2 + d_{1'}^2 + d_2^2 = 1 + 1 + 4 = 6 = r\,.$$

   Since $r = \sum_{a:irreducible\ representation}(d_a)^2$, we have confirmed that the irreducible representations of D_3 are $\rho^{(1)}$, $\rho^{(1')}$ and $\rho^{(2)}$ only.

4. By using the formula we obtained

$$m_a = \frac{1}{r}\sum_{i=1}^{n_c} r_i \chi(C_i)\overline{\chi^{(a)}(C_i)}\,,$$

   we can compute

$$\begin{aligned} m_1 &= \frac{1}{6}\left[1(3\times 1) + 2(0\times 1) + 3(1\times 1)\right] = 1\,, \\ m_{1'} &= \frac{1}{6}\left[1(3\times 1) + 2(0\times 1) + 3(1\times(-1))\right] = 0\,, \\ m_2 &= \frac{1}{6}\left[1(3\times 2) + 2(0\times(-1)) + 3(1\times 0)\right] = 1\,. \end{aligned}$$

   This result means that the representation ρ contains one $\rho^{(1)}$ and one $\rho^{(2)}$ as irreducible representations. This is consistent with what we have carried out by using matrices in Problem 1 of Problem Set No.2.

—-memo—-

For two elements in the same conjugacy class, the characters of a given representation take the same value. Let us consider a conjugacy class C_i (with a representative $g_i \in G$) of a finite group G which is defined as $C_i = \{ag_ia^{-1}|a \in G\}$. Then, for $g, h \in C_i$, there exist $a, b \in G$ such that $g = ag_ia^{-1}, h = bg_ib^{-1}$. Then we obtain $g_i = a^{-1}ga = b^{-1}hb$. Thus $g = ab^{-1}h(ab^{-1})^{-1}$. Now we consider a representation of G and denote matrix representation as $M(g)$ for $g \in G$. Then for g and h in the same conjugacy class as able, we have $M(g) = M((ab^{-1})h(ab^{-1})^{-1}) = M(ab^{-1})M(h)(M(ab^{-1}))^{-1}$. Thus by taking the trace, we obtain

$$\begin{aligned} \chi(g) &= tr(M(g)) = tr(M(ab^{-1})M(h)(M(ab^{-1}))^{-1}) = tr(M(h)(M(ab^{-1}))^{-1}M(ab^{-1})) \\ &= tr(M(h)) = \chi(h)\,. \end{aligned}$$

2 Induced Representation

1. What we need to show is

$$\sum_{l,b} M_{ia,lb}(g)M_{lb,kc}(g') = M_{ia,kc}(gg')\,. \tag{2.1}$$

Let us consider the following two cases separately: (1) $g_i^{-1}gg'g_k \in H$ (2) $g_i^{-1}gg'g_k \notin H$.

(1)Let us consider the first case: $g_i^{-1}gg'g_k \in H$. We first notice that $M_{ia,kc}(gg') = m_{ac}(g_i^{-1}gg'g_k)$.

Now we consider $g'g_k \in G$. Under the left coset decomposition of G, each element of G belongs to one and only one of g_lH's ($l = 1, 2, \cdots, n$). Therefore, there exits a unique j such that$g'g_k \in g_jH$. Then by multiplying g_j^{-1} from the left, we have $g_j^{-1}g'g_k \in H$. Since $g_i^{-1}gg'g_k \in H$, we have $g_i^{-1}gg_j = (g_i^{-1}gg'g_k)(g_j^{-1}g'g_k)^{-1} \in H$. On the other hand, from the definition of the right decomposition, we note that $g'g_k \notin g_lH$ for $l \neq j$ which leads to $g_l^{-1}g'g_k \notin H$ for $l \neq j$. By using these result we can obtain

$$\begin{aligned}\sum_{l,b} M_{ia,lb}(g)M_{lb,kc}(g') &= \sum_b M_{ia,jb}(g)M_{jb,kc}(g') \\ &= \sum_b m_{ab}(g_i^{-1}gg_j)m_{bc}(g_j^{-1}g'g_k) \\ &= m_{ac}(g_i^{-1}gg'g_k)\,.\end{aligned}$$

(2) Next we consider the case with $g_i^{-1}gg'g_k \notin H$. We first notice that $M_{ia,kc}(gg') = 0$.

If both $g_i^{-1}g'g_l \in H$ and $g_l^{-1}g'g_k \in H$ were satisfied, then we would have $g_i^{-1}gg'g_k = (g_i^{-1}g'g_l)(g_l^{-1}g'g_k) \in H$. This is inconsistent with the assumption. Therefore, at least one of $g_i^{-1}g'g_l$ and $g_l^{-1}g'g_k$ does not belong to H. Therefore $M_{ia,lb}(g)M_{lb,kc}(g') = 0$ for any l. Thus we have confirmed that $\sum_{l,b} M_{ia,lb}(g)M_{lb,kc}(g') = 0$.

To summarize, we have confirmed (2.1).

2. Let us start with

$$\chi_{ind}^{(\alpha)}(g) = \sum_{A:\text{irrep. of } G} n_A^{(\alpha)}\chi_G^{(A)}(g)\,.$$

By multiplying $\overline{\chi_G^{(B)}(g)}$ and then taking sum over $g \in G$, we obtain

$$\sum_{g\in G}\overline{\chi_G^{(B)}(g)}\chi_{ind}^{(\alpha)}(g) = \sum_{A:\text{irrep. of } G} n_A^{(\alpha)}\sum_{g\in G}\overline{\chi_G^{(B)}(g)}\chi_G^{(A)}(g) = p|H|n_B^{(\alpha)}\,.$$

Here we have used

$$\sum_{g\in G}\overline{\chi_G^{(B)}(g)}\chi_G^{(A)}(g) = |G|\delta_{AB} = p|H|\delta_{AB}\,.$$

On the other hand, since

$$\chi_{ind}^{(\alpha)}(g) = tr(M_{ind}^{(\alpha)}(g))$$

$$
\begin{aligned}
&= \sum_{i,a} M^{(\alpha)}_{ind,ia,ia}(g) \\
&= \sum_{a} \sum_{i:g_i^{-1}gg_i\in H} m^{(\alpha)}_{aa}(g_i^{-1}gg_i) \\
&= \sum_{i:g_i^{-1}gg_i\in H} \chi^{(\alpha)}(g_i^{-1}gg_i)\,.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\sum_{g\in G}\overline{\chi^{(B)}_G(g)}\chi^{(\alpha)}_{ind}(g) &= \sum_{g\in G}\overline{\chi^{(B)}_G(g)}\sum_{i:g_i^{-1}gg_i\in H}\chi^{(\alpha)}(g_i^{-1}gg_i) \\
&= \sum_{g\in G}\sum_{i:g_i^{-1}gg_i\in H}\overline{\chi^{(B)}_G(g)}\chi^{(\alpha)}(g_i^{-1}gg_i) \\
&= \sum_{g\in G}\sum_{i:g_i^{-1}gg_i\in H}\overline{\chi^{(B)}_G(g_i^{-1}gg_i)}\chi^{(\alpha)}(g_i^{-1}gg_i) \\
&= \sum_{i}\sum_{h\in H}\overline{\chi^{(B)}_G(h)}\chi^{(\alpha)}(h) \\
&= p\sum_{h\in H}\overline{\chi^{(B)}_G(h)}\chi^{(\alpha)}(h) \\
&= p\sum_{h\in H}\sum_{\beta:\text{irrep of } H} n^{(B)}_\beta\overline{\chi^{(\beta)}(h)}\chi^{(\alpha)}(h) \\
&= p\sum_{\beta:\text{irrep of } H} n^{(B)}_\beta\delta_{\alpha\beta}|H| \\
&= pn^{(B)}_\alpha|H|\,.
\end{aligned}
$$

In the middle we have used the orthogonality

$$
\sum_{h\in H}\overline{\chi^{(\beta)}(h)}\chi^{(\alpha)}(h) = |H|\delta_{\alpha\beta}\,.
$$

We also used the fact that for fixed i, for each element $h \in H$, there exists one and only one $g \in G$ such that $h = g_i^{-1}gg_i$. Therefore $\sum_{g\in G}\sum_{i:g_i^{-1}gg_i\in H}$ is replaced by $\sum_i\sum_{h\in H}$ and at the same time inside of the sum $g_i^{-1}gg_i$ is replaced by h.
Thus we have confirmed (by replacing B by A)

$$
n^{(\alpha)}_A = n^{(A)}_\alpha\,.
$$