Problem Set for Exercise Session No.5

Course: Mathematical Aspects of Symmetries in Physics,
ICFP Master Program (for M1)

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1 Some Basics on Manifolds

Answer the following questions:
(1) Here we confirm in two ways that a two-sphere with unit radius, $S^2 = \{(x,y,z) \in \mathbb{R}^3 | x^2+y^2+z^2=1\}$, is a two-dimensional differentiable manifold of class C^∞.

1. We take an atlas $\{(O_x^\pm, \varphi_x^\pm), (O_y^\pm, \varphi_y^\pm), (O_z^\pm, \varphi_z^\pm)\}$ where the open sets are defined by

$$
\begin{aligned}
O_x^+ &= \{(x,y,z) \in S^2 | x > 0\}\,, & O_x^- &= \{(x,y,z) \in S^2 | x < 0\}\,,\\
O_y^+ &= \{(x,y,z) \in S^2 | y > 0\}\,, & O_y^- &= \{(x,y,z) \in S^2 | y < 0\}\,,\\
O_z^+ &= \{(x,y,z) \in S^2 | z > 0\}\,, & O_z^- &= \{(x,y,z) \in S^2 | z < 0\}\,.
\end{aligned}
$$

   and $\varphi_x^\pm, \varphi_y^\pm, \varphi_z^\pm$ are maps from the corresponding open sets to a two-dimensional open disk $D^2 = \{(a,b) \in \mathbb{R}^2 | a^2+b^2 < 1\}$:

$$
\begin{aligned}
\varphi_x^+(x,y,z) &= \varphi_x^-(x,y,z) = (y,z)\,,\\
\varphi_y^+(x,y,z) &= \varphi_y^-(x,y,z) = (x,z)\,,\\
\varphi_z^+(x,y,z) &= \varphi_z^-(x,y,z) = (x,y)\,.
\end{aligned}
$$

   By using this atlas, compute the transition functions to confirm that S^2 is a two-dimensional differentiable manifold of class C^∞.

2. Let us consider another atlas $\{(U^\pm, f^\pm)\}$ for S^2 where the open sets are defined by

$$
U^\pm = S^2 \backslash \{(0,0,\pm 1)\}\,,
$$

   and $f^\pm$ are stereographic projections of S^2 from $(0,0,\pm 1)$ to (x,y)-plane. By using this atlas, confirm that S^2 is a two-dimensional differentiable manifold of class C^∞.

(2) Let us consider a two-dimensional real projective space $\mathbb{R}P^2$ defined as a quotient of $\mathbb{R}^3 \backslash \{(0,0,0)\}$ under the equivalence relation

$$
(x,y,z) \quad \sim \quad (x',y',z') \quad \Leftrightarrow \quad (x',y',z') = \lambda(x,y,z)\,, \qquad \lambda \in \mathbb{R}\backslash\{0\}\,,
$$

for $(x,y,z), (x',y',z') \in \mathbb{R}^3 \backslash \{(0,0,0)\}$. We denote the equivalence class for (x,y,z) by $[x:y:z]$.

1. We take an atlas $\{(U_x, \varphi_x), (U_y, \varphi_y), (U_z, \varphi_z)\}$ where the open sets are given by

   $$U_x = \{[x:y:z] \in \mathbb{R}P^2 | x \neq 0\}\,, \;\; U_y = \{[x:y:z] \in \mathbb{R}P^2 | y \neq 0\}\,, \;\; U_z = \{[x:y:z] \in \mathbb{R}P^2 | z \neq 0\}\,,$$

   and the maps from these open sets to $\mathbb{R}^2$ as

   $$\varphi_x([x:y:z]) = \left(\frac{y}{x},\, \frac{z}{x}\right)\,, \qquad \varphi_y([x:y:z]) = \left(\frac{x}{y},\, \frac{z}{y}\right)\,, \qquad \varphi_z([x:y:z]) = \left(\frac{x}{z},\, \frac{y}{z}\right)\,.$$

   Compute the transition functions to confirm that $\mathbb{R}P^2$ is a two-dimensional differentiable manifold of class C^∞.

2. Let us consider a quotient of S^2 by the equivalence relation

   $$(x, y, z) \quad \sim \quad (x', y', z') \quad \Leftrightarrow \quad (x', y', z') = \pm(x, y, z)\,,$$

   for $(x, y, z), (x', y', z') \in S^2$ (that is, identify the antipodal points). We denote this quotient of S^2 as $S^2/\sim$. Show that $\mathbb{R}P^2$ is homeomorphic to $S^2/\sim$.

2 Tangent Vector

Let us consider a point $p = (\sin\phi_0, 0, \cos\phi_0)$ (for $\phi_0 \in (0, \pi])$ on $S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$ and a curve $c : (-\epsilon, \epsilon) \to S^2$ (for $\epsilon > 0$) defined by

$$c(t) = (\sin\phi_0 \cos t, \sin\phi_0 \sin t, \cos\phi_0)\,.$$

We note that this curve satisfies $c(t = 0) = p$. Write down the tangent vector to this curve at p by using the coordinate introduced in Problem 1 (1)-2 through the stereographic projection from the north pole $(0, 0, 1) \in S^2$.

Note on Revision

Janurary 23 2015
Some problems in Problem 2 (in the old version) are moved to Problem Set No.7 and 8.