Problem Set for Exercise Session No.2

Course: Mathematical Aspects of Symmetries in Physics,
ICFP Master Program (for M1)

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1 Representation of D_3

Let us consider $\mathbf{R}^3$ (three-dimensional Euclidean space) and an equilateral triangle (with the length of the side $\sqrt{2}$) as in Fig.1. In this Figure, $\mathbf{e}_i$ is the unit vector along the i-th axis of the Cartesian coordinate ($i = 1, 2, 3$):

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} , \qquad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} .$$

These unit vectors form an orthonormal basis of the three-dimensional Euclidean space. Since elements of $D_3 = \{e, c_3, c_3^{-1}, \sigma_1, \sigma_2, \sigma_3\}$ by definition map this equilateral triangle into itself, we can write down the action of $g \in D_3$ as

$$g\,\mathbf{e}_i = \sum_{j=1}^{3} \mathbf{e}_j R_{ji}(g) \qquad (\text{for } \ i = 1, 2, 3) .$$

Therefore corresponding to each element $g \in D_3$, we can assign a 3×3 matrix $R(g) = (R_{ij}(g))$. Answer the following questions:

1. Show that $R(g)$'s are given by

$$R(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \qquad R(c_3) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} , \qquad R(c_3^{-1}) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} ,$$
$$R(\sigma_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \qquad R(\sigma_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \qquad R(\sigma_3) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{1.1}$$

   One can explicitly write down the multiplication table of $R(g)$'s to see that R is a representation matrix for a three-dimensional representation of D_3. We denote this representation as ρ.

2. Now we introduce another unit orthonormal basis

$$\tilde{\mathbf{e}}_1 = \frac{1}{\sqrt{3}} (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) , \qquad \tilde{\mathbf{e}}_2 = \frac{1}{\sqrt{6}} (2\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3) , \qquad \tilde{\mathbf{e}}_3 = \frac{1}{\sqrt{2}} (\mathbf{e}_2 - \mathbf{e}_3) .$$

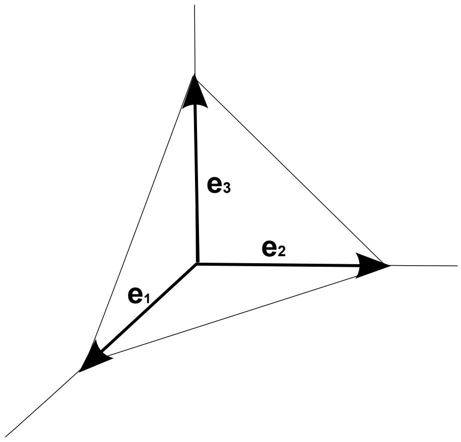

Figure 1: Equilateral triangle in $\mathbf{R}^3$.

By using this basis, we can similarly construct the representation matrix $\tilde{R}(g) = (\tilde{R}_{ij}(g))$ defined by

$$g\,\tilde{\mathbf{e}}_i = \sum_{j=1}^{3} \tilde{\mathbf{e}}_j \tilde{R}_{ji}(g) \qquad (\text{for } \ i = 1, 2, 3).$$

Show that $\tilde{R}(g)$'s are given by

$$\tilde{R}(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{R}(c_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad \tilde{R}(c_3^{-1}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix},$$

$$\tilde{R}(\sigma_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \tilde{R}(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \quad \tilde{R}(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}.$$

We can check that there is no basis which diagonalizes all the $R(g)$'s. This expression indicates that this representation ρ is completely reducible, and the 1×1 block and the 2×2 block in $\tilde{R}(g)$'s

$$R^{(1)}(g) = 1\,, \qquad (\text{for } \ \forall g \in D_3),$$

$$R^{(2)}(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R^{(2)}(c_3) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad R^{(2)}(c_3^{-1}) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix},$$

$$R^{(2)}(\sigma_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad R^{(2)}(\sigma_2) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \quad R^{(2)}(\sigma_3) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

are representation matrices of a one-dimensional irreducible representation (denoted as $\rho^{(1)}$) and a two-dimensional irreducible representation (denoted as $\rho^{(2)}$).

3. Are $\rho^{(1)}$ and $\rho^{(2)}$ unitary?

4. Let us consider another 1×1 matrices $R^{(1')}(g)$:

$$R^{(1')}(e) = R^{(1')}(c_3) = R^{(1')}(c_3^{-1}) = 1\,, \qquad R^{(1')}(\sigma_1) = R^{(1')}(\sigma_2) = R^{(1')}(\sigma_3) = -1\,.$$

Write down the multiplication table of $R^{(1')}(g)$'s to confirm that $R^{(1')}(g)$ is indeed a representation matrix for a one-dimensional irreducible representation of D_3. We call this representation as $\rho^{(1')}$.