Homework 1

* each $\hat{X}_i$ falls in $[u,v]$, independently on the others, with probability

$$\mathbb{P}\left[\hat{X}\in[u,v]\right] = \mathbb{P}\left[X\in[a_n+b_n u, a_n+b_n v]\right] = F_X(a_n+b_n v) - F_X(a_n+b_n u) \equiv p_n([u,v])$$

hence $N_n([u,v]) \overset{d}{=} \mathrm{Bin}\left(n, p_n([u,v])\right)$

when $n\to\infty$, $p_n([u,v]) = \frac{1}{n}\left(\gamma(u)-\gamma(v)\right) + o\left(\frac{1}{n}\right) \Rightarrow N([u,v]) \overset{d}{=} \mathcal{P}o\left(\gamma(u)-\gamma(v)\right)$

* independently on each other, each $\hat{X}_i$ falls in $[u_1,v_1]$ with proba $p_n([u_1,v_1])$

$[u_2,v_2]$ $\quad p_n([u_2,v_2])$

...

$[u_p,v_p]$ $\quad p_n([u_p,v_p])$

hence $\left(N_n([u_1,v_1]),\dots,N_n([u_p,v_p])\right) \overset{d}{=} \mathrm{Multinomial}\left(n, p_n([u_1,v_1]),\dots,p_n([u_p,v_p])\right)$

ie
$$\mathbb{P}\left[N_n([u_1,v_1])=k_1,\dots,N_n([u_p,v_p])=k_p\right] = \frac{n!}{k_1!\dots k_p!(n-k_1-\dots k_p)!}\, p_n([u_1,v_1])^{k_1}\dots p_n([u_p,v_p])^{k_p} \times$$
$$\left(1-p_n([u_1,v_1])-\dots-p_n([u_p,v_p])\right)^{n-k_1-\dots-k_p}$$

$$\xrightarrow[n\to\infty]{} e^{-(\gamma(u_1)-\gamma(v_1))}\frac{(\gamma(u_1)-\gamma(v_1))^{k_1}}{k_1!}\dots e^{-(\gamma(u_p)-\gamma(v_p))}\frac{(\gamma(u_p)-\gamma(v_p))^{k_p}}{k_p!}$$

in the limit the $N([u,v])$ are independent Poisson random variables :

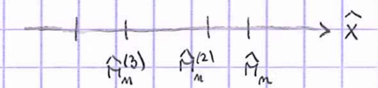
the $\{\hat{X}_i\}$ form a Poisson Point Process, with intensity measure $\mu([u,v]) = \gamma(u)-\gamma(v)$

where γ is one of the three universal extreme value distribution

* link with the question : $\mathbb{P}\left[\hat{M}_n\le x\right] = \mathbb{P}\left[N_n([x,\infty[)=0\right] \xrightarrow[n\to\infty]{} \mathbb{P}\left[\mathcal{P}o(\mu([x,\infty[)=0\right]$

$= e^{-\gamma(x)}$ same result as in TD

call $\hat{M}_n^{(2)}$ the second largest



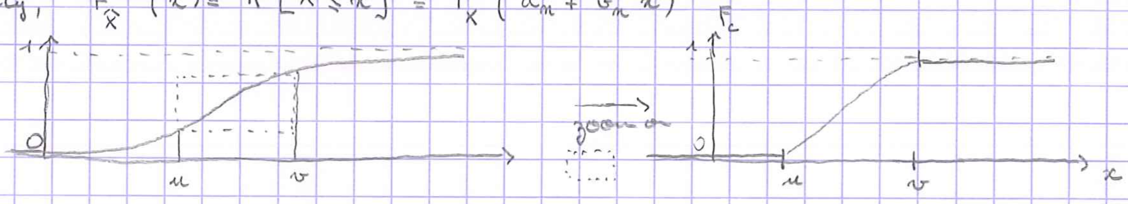
$$\mathbb{P}\left[\hat{M}_n^{(2)}\le x\right] = \mathbb{P}\left[N_n([x,\infty[)=0 \text{ or } 1\right] \to \mathbb{P}\left[\mathcal{P}o(\mu([x,\infty[))=0 \text{ or } 1\right]$$

$$= e^{-\gamma(x)}\left(1+\gamma(x)\right)$$

in general, $\mathbb{P}\left[\hat{M}_n^{(k)}\le x\right] \to \mathbb{P}\left[\mathcal{P}o\left(\mu[x,\infty[\right)\le k-1\right] = e^{-\gamma(x)}\left(1+\gamma(x)+\frac{1}{2!}\gamma(x)^2+\dots+\frac{1}{(k-1)!}\gamma(x)^{k-1}\right)$

* at finite n, the $\hat{X}_i$ are iid with some measure μ $\to$ conditional on $N_n([u,v]) = p$, the p points in this interval are iid with $\frac{\mu \, \mathbb{1}_{[u,v]}}{\mu([u,v])}$, the measure conditioned on the interval

more precisely, $F_{\hat{X}}^{(n)}(x) = \mathbb{P}[\hat{X} \leq x] = F_X(a_n + b_n x)$



conditional distribution : $F_c(x) = \begin{cases} 0 & \text{if } x \leq u \\ 1 & \text{if } x \geq v \\ \dfrac{F_{\hat{X}}^{(n)}(x) - F_{\hat{X}}^{(n)}(u)}{F_{\hat{X}}^{(n)}(v) - F_{\hat{X}}^{(n)}(u)} & \text{for } x \in [u,v] \end{cases}$

when $n \to \infty$, $F_{\hat{X}}^{(n)}(x) = 1 - \frac{1}{n}\gamma(x) + o\left(\frac{1}{n}\right)$

$\Rightarrow$ $F_c(x) = \dfrac{\gamma(u) - \gamma(x)}{\gamma(u) - \gamma(v)}$ on $[u,v]$

$\Rightarrow$ in the limit, the p points are iid on $[u,v]$ with density $\dfrac{-\gamma'(x)}{\gamma(u) - \gamma(v)}$

check : u u' v' v one can contruct $N([u',v'])$ by :

- drawing $N([u,v]) \to \mathcal{P}o(\gamma(u) - \gamma(v))$
- conditional on $N([u,v])$ denoted p, draw the points with the above density, each fall in $[u',v']$ with proba $\dfrac{\gamma(u') - \gamma(v')}{\gamma(u) - \gamma(v)}$ $\Rightarrow$ $\mathrm{Bin}\left(p, \dfrac{\gamma(u') - \gamma(v')}{\gamma(u) - \gamma(v)}\right)$

hence $\mathbb{P}[N([u',v']) = p'] = \displaystyle\sum_{p=p'}^{\infty} e^{-(\gamma(u) - \gamma(v))} \frac{(\gamma(u) - \gamma(v))^p}{p!} \binom{p}{p'} \left(\frac{\gamma(u') - \gamma(v')}{\gamma(u) - \gamma(v)}\right)^{p'} \left(1 - \frac{\gamma(u') - \gamma(v')}{\gamma(u) - \gamma(v)}\right)^{p-p'}$

$$= \frac{1}{p'!} e^{-(\gamma(u) - \gamma(v))} (\gamma(u') - \gamma(v'))^{p'} \underbrace{\sum_{p=p'}^{\infty} \frac{1}{(p-p')!} (\gamma(u) - \gamma(v))^{p-p'} \left(1 - \frac{\gamma(u') - \gamma(v')}{\gamma(u) - \gamma(v)}\right)^{p-p'}}_{e^{\gamma(u) - \gamma(v) - (\gamma(u') - \gamma(v'))}}$$

$$= e^{-(\gamma(u') - \gamma(v'))} \frac{(\gamma(u') - \gamma(v'))^{p'}}{p'!}$$

this is indeed the expected Poisson law, which justifies the conditional distribution of the points