ICFP M2 - Statistical Physics: Advanced and New Applications – Exam

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The exam is made of two parts. The first one is a series of short independent questions to check your knowledge and understanding of the contents of some of the lectures. You are expected to provide concise answers, without long computations. The second one consists in two longer independent problems.

Please write your answers to the two parts on separate pages.

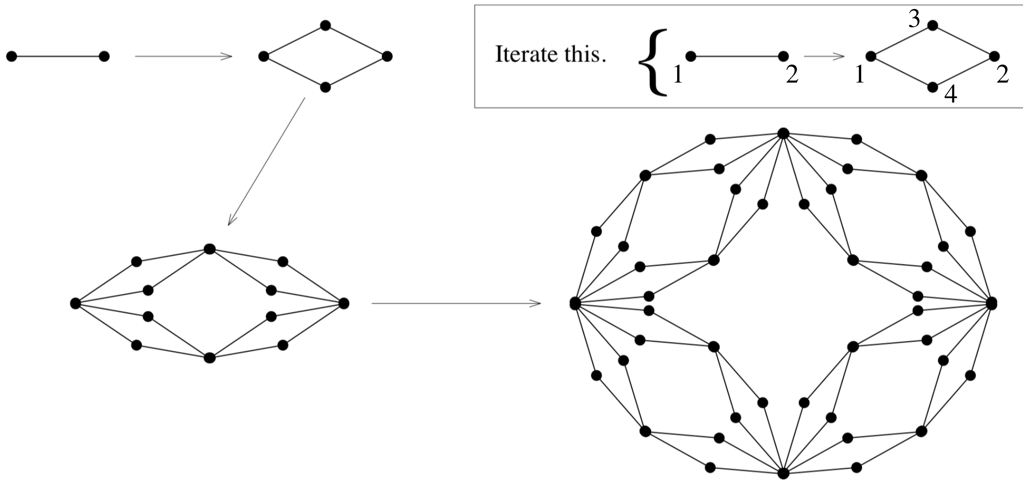
No document nor calculator is allowed.

1 Questions on the lectures ($\approx 3/18$)

1. Give the definition of the mutual information and recall its meaning.
2. What is the upper critical dimension of a second order phase transition ?
3. What are the identities that can be derived using the time-reversal symmetry of equilibrium dynamics ?

2 Ising model on the Berker lattice ($\approx 6/18$)

The Berker lattice is a hierarchical lattice, defined in an iterative way as follows



i.e. the lattice is made by repeatedly replacing every edge by the quadrilateral shown. We consider a spin system where Ising spins $\sigma_i = \pm 1$ are located on the sites (the black dots) of this lattice (obtained after a large number of iterations $r \gg 1$). The lines connecting two sites correspond to ferromagnetic bonds. The Hamiltonian H, in presence of a magnetic field $\overline{h}$ is given by

$$\beta H = -\beta \overline{J} \sum_{\langle i,j \rangle} \sigma_i \sigma_j - \beta \overline{h} \sum_i \sigma_i \ , \tag{1}$$

where $\overline{J} > 0$ is the ferromagnetic coupling, and $\langle i, j \rangle$ denotes nearest neighbours (connected by a bond) on the Berker lattice. Henceforth, we will use the notation $J = \beta \overline{J}$ and $h = \beta \overline{h}$.

1. What is the number of bonds after r iterations ? Using this result, show, by induction on r, that the total number of sites after r generations is $N_r = (2 \times 4^r + 4)/3$. Argue that the typical linear size L of the Berker lattice (measured in steps on the lattice) is $L \sim 2^r$ and explain why the effective dimensionality of the system is $d = 2$.

   We want to perform a real space renormalization group (RG) scheme by decimating the spins which are connected only to two neighbours.
2. Show that the scale factor of this RG transformation is $b = 2$.
3. Explain why this procedure can be performed exactly on the Berker lattice by simply considering the partition function of the system with simply 4 sites (see the figure above) and rewriting it as

$$Z_{\Diamond} = \sum_{\sigma_1 = \pm 1, \sigma_2 = \pm 1} e^{J' \sigma_1 \sigma_2 + h'(\sigma_1 + \sigma_2) + g'} \tag{2}$$

   where the renormalised couplings (J', h') are given by

$$J' = \frac{1}{2} \ln \left(\frac{\cosh(2J + h) \cosh(2J - h)}{\cosh^2 h} \right) \tag{3}$$

$$h' = h + \frac{1}{2} \ln \left(\frac{\cosh(2J + h)}{\cosh(2J - h)} \right) \ . \tag{4}$$

4. Consider first the case $h = 0$ and study (graphically) the fixed points of the renormalisation group transformation for the coupling J in Eq. (3) together with their stability.
5. Recall the renormalization group transformation for the $1d$ Ising model (on a regular lattice) with $h = 0$ and compare the two systems ($1d$ versus Berker lattice).
6. We denote by J^* the *critical* coupling of the Ising model on the Berker lattice in the absence of magnetic field. By linearizing the RG equation (3) – setting $h = 0$ – around J^*, compute the critical exponent ν in terms of J^*.
7. Linearize the RG equations (3) and (4) for $0 < h \ll 1$ and sketch the RG flow in the (J, h) plane.

3 Arrhenius law out of equilibrium ($\approx 9/18$)

The Arrhenius law plays a central role in several different contexts, e.g. for metastability, chemical reaction rates, conductivity in semi-conductors, viscosity of glass-forming liquids, etc.
It is related to the crossing of an energetic barrier, which at low temperature is a very rare process. The Arrhenius law gives the typical time-scale on which the barrier is crossed as a function of the height of the barrier, $\Delta = V_{barrier} - V_{minimum}$, and the temperature as $\tau \simeq \tau_0(T)e^{\Delta/T}$ (V is the configurational energy, or the potential). The Arrhenius law is valid in the regime $T \ll \Delta$.
In the following we shall show that in out of equilibrium cases in which the motion is induced by non-conservative forces, i.e. which do not derive from a potential, one can find a generalization of the Arrhenius law. The analysis will be performed in a simplified model defined by the Langevin equation :

$$\frac{d\mathbf{x}}{dt} = \mathbf{F} + \boldsymbol{\eta} \tag{5}$$

where $\mathbf{F}$ is the force field, $\boldsymbol{\eta}$ a Gaussian noise, corresponding to a thermal bath (symbols in bold denote vectors : $\mathbf{x} = \{x_\alpha\}$ for $\alpha = 1, ..., d$). We assume that $\boldsymbol{\eta}$ has zero mean and variance $\langle \eta_\alpha(t)\eta_\beta(t')\rangle = 2T\delta(t-t')\delta_{\alpha,\beta}$ where T is the strength of the thermal fluctuations. The force field $\mathbf{F}$ is non-conservative, i.e. it cannot be written as the gradient of a potential, therefore the system is kept out of equilibrium : it dissipates heat with the reservoir and does (or receives) work because of the force field.
We further assume that the force field $\mathbf{F}$ has two stable points S_1, S_2, one unstable point U and no limit-cycles. At zero temperature, all points $\mathbf{x}$ can be divided in three sets : the basin of attraction of S_1, i.e. all the initial conditions for the dynamics $\frac{d\mathbf{x}}{dt} = \mathbf{F}$ from which at infinite time the system reaches S_1, the basin of attraction of S_2, and the point U which is an unstable stationary point. A two-dimensional example is provided in Fig. 1.
In the following we shall consider the low noise limit, $T \ll 1$, and study the rare process : $S_1 \to U \to S_2$ which is the out of equilibrium counterpart of barrier crossing (note however that now a priori there is no barrier since there is no potential !).

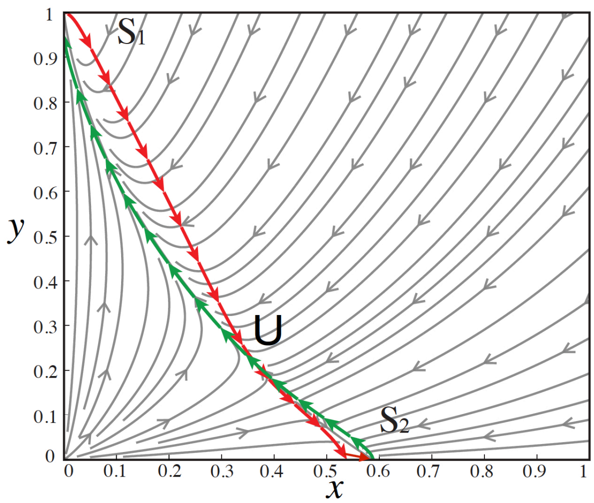


FIGURE 1 – Two-dimensional force field $\mathbf{F}(x,y) = (\frac{x}{x+fy} - \epsilon x, \frac{fy}{x+fy} - \psi xy - y)$ plotted with gray lines ($\epsilon = 1.7$, $f = 1.12$, $\psi = 1.1$). Saddle point trajectories of $\mathbf{x}(t)$ in the force field $\mathbf{F}(x,y)$ are shown by red and green arrows.

1. Show that the Fokker-Planck equation for the probability density of $\mathbf{x}$ reads :

$$\frac{\partial}{\partial t}P(\mathbf{x},t) = \frac{\partial}{\partial \mathbf{x}}\left[-\mathbf{F} + T\frac{\partial}{\partial \mathbf{x}}\right]P(\mathbf{x},t)$$

2. Writing the stationary probability measure as $P_s = e^{-\phi/T}/Z$, where Z is a normalization factor,

show that in the $T \to 0$ limit ϕ verifies the equation :

$$\boldsymbol{\nabla}\phi \cdot (\boldsymbol{F} + \boldsymbol{\nabla}\phi) + \mathcal{O}(T) = 0 \tag{6}$$

In the equilibrium case, where $\mathbf{F} = -\boldsymbol{\nabla} V$, what is ϕ equal to ?

3. Write down the Martin-Siggia-Rose-DeDominicis-Janssen action corresponding to the stochastic equation (5) [following the Ito's convention for the underlying discretisation]. Use the notation $\hat{\mathbf{x}}$ for the imaginary response field (Do not repeat the derivation, just write down the action explaining why the derivation we did in class straightforwardly generalizes to this case).
4. Make a change of variable in the functional integral, $\hat{\mathbf{x}}_{new} = \hat{\mathbf{x}}/T$, and justify why the sum over paths is dominated by a saddle point contribution in the $T \to 0$ limit. The purpose of changing variables is only to justify the saddle-point, in the following we will keep using $\hat{\mathbf{x}}$ and not $\hat{\mathbf{x}}_{new}$.
5. Obtain the saddle point equations which read :

$$\begin{aligned} \frac{\delta S}{\delta x_\alpha} = 0 : \quad & \frac{d\hat{x}_\alpha}{dt} = -\sum_\beta \left[\frac{\partial F_\beta}{\partial x_\alpha} \hat{x}_\beta \right] \\ \frac{\delta S}{\delta \hat{x}_\alpha} = 0 : \quad & \frac{dx_\alpha}{dt} - F_\alpha = 2T\hat{x}_\alpha \end{aligned} \tag{7}$$

6. Show that

$$\frac{d\mathbf{x}}{dt} = \mathbf{F} \qquad \hat{\mathbf{x}} = 0\,,$$

   is a solution of the saddle-point equations, which we will call P_1.
7. Show that

$$\frac{d\mathbf{x}}{dt} = \mathbf{F} + 2\boldsymbol{\nabla}\phi \qquad \hat{\mathbf{x}} = \boldsymbol{\nabla}\phi/T\,, \tag{8}$$

   is a solution of the saddle-point equations, which we will call P_2. (suggestion : it can be checked by direct substitution and by using the derivative of eq. (6) ; ϕ was defined at question (2).)
8. U is an unstable point and a system starting infinitely close to U can reach either S_1 or S_2 (depending on the initial point) following the force field. This is what happens exactly at zero temperature, and corresponds to the solution of the equation $\frac{d\mathbf{x}}{dt} = \mathbf{F}$. The trajectory $T_{U\to S_2}$ followed by the system is generically obtained solving the previous equation with initial condition in U and arrival at infinite time in S_2.
   In consequence, the process $U \to S_2$ corresponds to the solution P_1. Evaluate its action, and deduce that it happens with finite probability in the $T \to 0$ limit, and explain why.
9. The process $S_1 \to U$ is instead very improbable in the $T \to 0$ limit, as it can be expected on general grounds since the system has to move against the force field. It can be shown that there is a unique saddle point solution P_2 that corresponds to it, which has initial condition in S_1 and arrival at infinite times in U.
   Compute the action of such solution (suggestion : make use of eq. (6) to simplify the expression). Show that at leading order, the probability of the process $S_1 \to U$ is $e^{-\Delta\phi/T}$ where $\Delta\phi = \phi(\mathbf{x}_U) - \phi(\mathbf{x}_{S_1})$.
10. The process $S_1 \to S_2$ can be obtained by patching together $S_1 \to U$ and $U \to S_2$. Using the previous results, argue why the timescale for the process $S_1 \to S_2$ is proportional to $e^{\Delta\phi/T}$, which is the Arrhenius law out of equilibrium.
11. Consider now the equilibrium case, $\mathbf{F} = -\boldsymbol{\nabla} V$, and show that the solution P_2 is the time-reversed of the solution P_1. Explain why this is the case at equilibrium. Show that at variance with the out of equilibrium case this relation this relation between P_1 and P_2 does not hold.