ICFP M2 - Statistical physics 2 – Exam
Hysteresis in the Random Field Ising Model

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The exam is made of two parts. The first one is a series of short independent questions to check your knowledge and understanding of the contents of some of the lectures. You are expected to provide concise answers, without long computations. The second one is a longer problem, divided in three parts which are partially independent.

Please write your answers to the two parts on separate pages.

No document nor calculator is allowed.

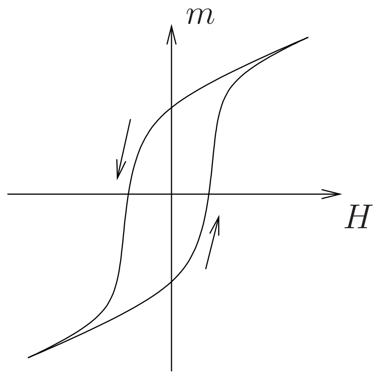
1 Questions on the lectures ($\approx 6/20$)

1. Give the definition of the quenched and annealed free-energy density for a disordered model with Hamiltonian $H(\underline{\sigma}; \underline{J})$, where $\underline{\sigma}$ are discrete degrees of freedom and $\underline{J}$ some quenched disorder variables. Recall the inequality between these two free-energies and its derivation.
2. The Inverse Participation Ratio of an N-dimensional vector v is defined as IPR $= \sum_{i=1}^{N} |v_i|^4$, assuming that v is normalized with $\sum_{i=1}^{N} |v_i|^2 = 1$. Recall the behavior of IPR at large N when v is "localized" and when v is "extended".
3. What is the order parameter which changes from 0 to a positive value at the percolation phase transition ? What is the mean-field value of the critical exponent describing this behavior ?
4. We consider a random variable X which takes values in $[0, 1]$ and which has the density $f_X(x) = 2x$ on this interval. We define $S_n = X_1 + \cdots + X_n$ and $M_n = \max(X_1, \ldots, X_n)$ as the sum and the maximum of $X_1, \ldots, X_n$ which are n independent copies of the random variable X. Describe the scalings at large n of S_n and M_n.

2 Hysteresis in the mean-field Random Field Ising Model ($\approx 14/20$)

2.1 Introduction and notations

Many materials exhibit "hysteresis" at low temperature, i.e. their response to a modification of their external parameters is delayed in time. For instance this picture represents (in a schematic way) the response of the magnetization m of a ferromagnet when the external field H is varied cyclically. One sees that for some values of H the magnetization depends whether H is increasing or decreasing ; this shows that hysteresis is an out-of-equilibrium phenomenon, as the state of the system at a given time depends on its history.



We will study this phenomenon in the the Random Field Ising Model on a completely connected lattice, which provides a mean-field description of hysteresis. This is a disordered system with N Ising spins, their configurations being denoted $\underline{\sigma} = (\sigma_1, \dots, \sigma_N) \in \{-1, 1\}^N$. The energy of a configuration is

$$E(\underline{\sigma}; H, \underline{h}) = -\frac{1}{2N} \sum_{i \neq j} \sigma_i \sigma_j - \sum_{i=1}^{N} \sigma_i (h_i + H) \ ,$$

where H is the average external magnetic field and the quenched disorder $\underline{h} = (h_1, \dots, h_N)$ corresponds to site-dependent external magnetic fields. The h_i are independent random variables, each drawn from the probability density $\rho(h)$ assumed to be symmetric (hence of zero average).

We define the hysteresis loop of the model by a quasistatic (or adiabatic) protocol at zero temperature : we start with $H = -\infty$, and denote $\underline{\sigma}^-(-\infty; \underline{h}) = (-1, \dots, -1)$ the configuration with all spins pointing down. We then increase very slowly the magnetic field H and define the sequence of configurations $\underline{\sigma}^-(H; \underline{h})$ by flipping spin i as soon as this flip decreases the energy of the configuration. For all values of H the configuration $\underline{\sigma}^-(H; \underline{h})$ is thus a local minimum of $E(\underline{\sigma}; H, \underline{h})$, its energy cannot be further decreased by flipping any single spin. When H becomes very large and positive we reach the configuration $\underline{\sigma}^-(+\infty; \underline{h}) = (+1, \dots, +1)$. The sequence $\underline{\sigma}^+(H; \underline{h})$ is obtained similarly by decreasing the field H from very large values, starting with $\underline{\sigma}^+(+\infty; \underline{h}) = (+1, \dots, +1)$ and flipping spins as soon as the energy can be decreased.

As emphasized by the notation the sequences of configurations $\underline{\sigma}^-(H; \underline{h})$ and $\underline{\sigma}^+(H; \underline{h})$ depend on the realization of the quenched disorder $\underline{h}$. We shall however see that in the thermodynamic limit their magnetizations

$$m^-(H; \underline{h}) = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^- \ , \qquad \text{and} \quad m^+(H; \underline{h}) = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^+$$

are self-averaging quantities.

We will use the notation $\text{sign}(x) = \begin{cases} 1 & \text{if } \ x > 0 \\ -1 & \text{if } \ x < 0 \end{cases}$.

2.2 Self-consistent equation for the hysteresis loop in the thermodynamic limit

1. We define the effective field acting on spin i in a configuration $\underline{\sigma}$ for an external field H and a quenched disorder $\underline{h}$ by

$$H_i(\underline{\sigma}; H, \underline{h}) = \left(\frac{1}{N} \sum_{j (\neq i)} \sigma_j \right) + h_i + H \ .$$

Compute the change in energy when spin σ_i is flipped from σ_i to $-\sigma_i$ starting from $\underline{\sigma}$, and express it in terms of the local field H_i.

Since we focus on the thermodynamic limit in the following we will neglect that the sum does not run over i and approximate the effective field by

$$H_i(\underline{\sigma}; H, \underline{h}) = \left(\frac{1}{N} \sum_{j=1}^{N} \sigma_j \right) + h_i + H \ .$$

2. Justify briefly the two statements :
   - Along the sequence of configurations $\underline{\sigma}^-(H; \underline{h})$ encountered while increasing H, all the spins are of the sign of their effective field.
   - Along this sequence of configurations, all spin flips are from -1 to $+1$.

   Similarly along $\underline{\sigma}^+(H; \underline{h})$ for decreasing H spins flip from $+1$ to -1.

3. We first consider the pure case in which all the random fields h_i are zero. What is the value of H at which all spins flip from -1 to $+1$ when H is increasing ? Draw the magnetizations $m^-(H)$ and $m^+(H)$ as a function of H along the hysteresis loop.

4. We come back to the disordered case, i.e. with $h_i \neq 0$. Consider a configuration $\underline{\sigma}$ which is a local minimum of the energy for a given value of H and of the quenched disorder $\underline{h}$. Show that the magnetization of this configuration, $m(H;\underline{h})$, satisfies a self-consistent equation. Take then the thermodynamic limit, assuming that the magnetization has small fluctuations around the limit of its mean, denoted $m(H)$, and show that this quantity is a solution of :

$$m(H) = \langle \mathrm{sign}(m(H) + h + H) \rangle \ ,$$

where the average is over the probability density $\rho(h)$ of the random fields.

5. Show that the previous equation can be recast in the form :

$$m(H) = R\left(m(H) + H\right) \ , \qquad \text{with} \qquad R(x) = -1 + 2\int_{-\infty}^{x} \rho(h)\,\mathrm{d}h \ , \tag{1}$$

exploiting the symmetry $\rho(h) = \rho(-h)$.

From now on we assume that the probability density of the random fields is $\rho(h) = e^{-|h|/\Delta}/(2\Delta)$.

6. Draw the shape of $R(x)$ as a function of x ; you should in particular study its limit in $\pm\infty$, its symmetry properties, the value of its derivative in 0, and the monotonicity of its derivative.
7. Solve graphically the equation (1) for $\Delta > 1$. Show that it admits only one solution for any value of H. Draw $m(H)$ qualitatively.
8. Solve graphically the equation (1) for $\Delta < 1$. Show that there exists values of H such that it admits more than one solution.

In the end of this part and in the next one we concentrate on the case $\Delta < 1$.

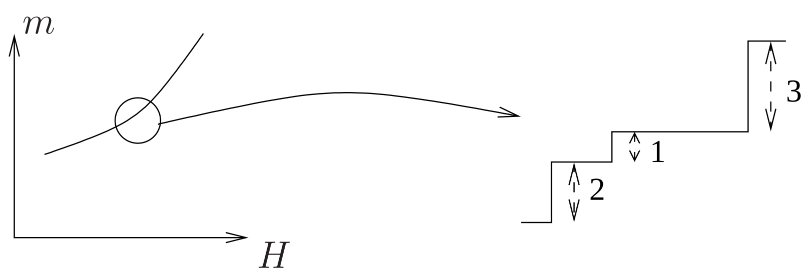
9. When changing H from $-\infty$ to $+\infty$ the magnetization $m^-(H)$ followed by the quasistatic protocol is the smallest solution of (1). Similarly when changing H from $+\infty$ to $-\infty$ the physical solution $m^+(H)$ is the largest solution of (1). Using these properties draw $m^-(H)$ and $m^+(H)$ qualitatively for $\Delta < 1$ and show that the lower branch of the hysteresis loop is the mirror image of the upper half : $m^-(H) = -m^+(-H)$.
10. Calling H^* the value at which the solution with the lowest value of m has a singularity and m^* its corresponding value, show graphically that at H^*, m^* :

$$1 = 2\,\rho(m^* + H^*) \tag{2}$$

11. Using equations (1,2) and the explicit form of $\rho(h)$ show that $m^* = \Delta - 1$ and $H^* = \Delta \log \Delta + (1 - \Delta)$.
12. Considering the lowest branch of the hysteresis cycle and $H = H^* - \epsilon$ with $\epsilon \ll 1$, show that $|m^* - m^-(H - \epsilon)| \propto \sqrt{\epsilon}$.

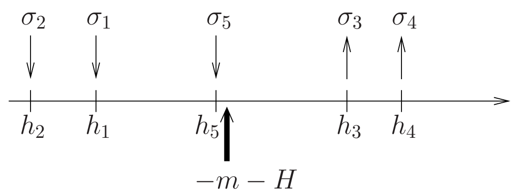
2.3 Avalanches and critical behavior

In the previous part we obtained the curves $m^-(H)$ and $m^+(H)$ as the thermodynamic limit of $m^-(H;\underline{h})$ and $m^+(H;\underline{h})$, for typical realizations of the quenched disorder. We shall now consider large but finite samples, and zoom in on the evolution of $m^-(H;\underline{h})$:



For a finite system the magnetization can only change by integer multiples of $2/N$, as shown on the right of the figure. This integer is called the size of the corresponding "avalanche" : the increase of the external magnetic field H first triggers the flip of one spin from -1 to $+1$ when its effective field becomes positive. This flip increases the effective field of all the other spins (without further modifying H). In consequence, some of them might acquire a positive effective field and thus flip too. This process, called avalanche, continues until a stable configuration is reached. In this part we shall study the probability distribution of the avalanche sizes along the hysteresis cycle.

1. In the figure below we represent a configuration of a small system with 5 spins ; the position on the axis denotes the value of the quenched disordered fields h_i, at a given point during the quasistatic increase of H :



   On the basis of the figure, explain why :
   — the configuration is stable (i.e. it is a local minimum of the energy).
   — an avalanche is about to start at spin 5.

2. Explain, with the help of a similar figure, the condition that ensures that the avalanche is of size 1.
3. Same question for avalanches of size 2.
4. A detailed computation, that you will not do, shows that an avalanche which starts in a configuration of magnetization m when the external magnetic field crosses the value H is of size s with probability

$$P(s) = \frac{(s\lambda)^{s-1}}{s!} e^{-s\lambda} \ , \qquad \text{with} \qquad \lambda = 2\,\rho(m+H) \ .$$

   By using the Stirling approximation $s! \simeq \sqrt{2\pi s}\left(\frac{s}{e}\right)^s$ show that the tail of $P(s)$ for large s is : $P(s) \simeq \frac{1}{\lambda}\frac{1}{\sqrt{2\pi s^3}} e^{s(1-\lambda+\ln\lambda)}$

5. Discuss how the tail of $P(s)$ changes when H increases along the branch $m^-(H)$ of the hysteresis loop. What happens when $m^-(H)$ is singular, i.e. at $H = H^*$?
6. Explain why the system presents a critical behavior similar to the one of standard second order phase transitions when $H \uparrow H^*$ along the branch $m^-(H)$.
7. We consider now a value of H so close to H^* that we can take $\lambda = 1$. We define the increase in the magnetization due to a large number M of independent avalanches as

$$\delta m = \frac{2}{N}\sum_{j=1}^{M} s_j \ ,$$

   where the s_j are i.i.d. random variables drawn from $P(s)$. How does δm scales with $M \gg 1$? What is the scaling of the largest of such avalanches and how does it compare to the entire sum ? An independent argument shows that M scales in the thermodynamic limit as $N^{1/3}$. Translate your results in terms of N.

2.4 Dynamical behavior

We now consider the dynamical behavior associated to a sudden change in the external field. We shall focus on a protocol in which the external field is suddenly switched from $H = -\infty$ to H_f at time $t = 0$. In the initial configuration all spins are -1. We model the subsequent off-equilibrium zero temperature dynamics in the following way : any given spin tries to flip in the interval of time $t, t+dt$ with probability $dt \ll 1$; if the process corresponds to a decrease in energy then the flip occurs, otherwise it does not.

1. We denote $m_i(t) = [\sigma_i(t)]$ the average magnetization of σ_i at time t, where the average is only over the randomness of the stochastic process. List all the possible events affecting σ_i between t and $t + dt$ and express $m_i(t + dt)$ as a function of the configuration at time t and then average over the stochastic process until time t. Use this result to show that

$$\frac{\mathrm{d}m_i(t)}{\mathrm{d}t} = -m_i(t) + \mathrm{sign}(m(t) + h_i + H_f) \ , \qquad \text{with} \quad m(t) = \frac{1}{N}\sum_{i=1}^{N} m_i(t) \ .$$

   In obtaining this equation one has to admit, as before, that $m(t)$ does not fluctuate around its mean.

2. Obtain the dynamical equation for the magnetization $m(t)$ in the thermodynamic limit, and show that $m(\infty)$ verifies the static equation obtained previously. This is a peculiarity of the Random Field Ising Model : the final configuration does not depend on the protocol (sudden or adiabatic) followed during the hysteresis cycle.
3. Show that as long as $H_f < H^*$ the magnetization $m(t)$ converges to $m(\infty)$ exponentially fast in time.
4. How does the magnetization $m(t)$ converges to $m(\infty)$ as a function of t when $H_f = H^*$?

2.5 Conclusion

Skip it during the exam, read it after. As discussed in the introduction, hysteresis takes place in many different situations, e.g. in magnetic materials due to a change of magnetic field, in shape-memory alloys due to strain deformations, in bacteria communities due to antibiotic treatments and in social sciences due to external perturbations. The random field Ising model provides a natural setting to study this phenomenon and serves as guideline in many contexts. See Refs [1,2] for reviews.
The transition that we studied in this exam–the discontinuous jump along the hysteresis loop–is an example of phase transition out of equilibrium which shares many properties with equilibrium ones, e.g. power law scaling at the critical point, singular behavior of the order parameter approaching the transition, etc.. In this exam we focused on the mean-field theory. Finite dimensional fluctuations lead to important qualitative changes, as it happens for equilibrium critical phenomena. Developing a complete theory valid in three dimension is a current research topic.

[1] J.P. Sethna, K.A. Dahmen, O. Perkovic, *Random-Field Ising Models of Hysteresis* in "The Science of Hysteresis Vol. II", arXiv : cond-mat/0406320.
[2] J.-P. Bouchaud, *Crises and collective socio-economic phenomena : simple models and challenges*, J. Stat. Phys **151**, 567 (2013), arXiv : 1209.0453.