Master ENS ICFP first year

Relativistic Quantum Mechanics and Introduction to Quantum Field Theory

Final exam : January 17, 2018 – duration : 4 hours

• The different problems are completely independent.

• Problems 1, 2 and 4 are related to the first part of the lectures ”Relativistic Quantum Mechanics”, and problems 3 and 5 to the second part of the lectures ”Introduction to Quantum Field Theory”.

• The number of points indicated for each problem is just an approximative indication of the importance of the problem. The final weight of each may still be adjusted differently.

• Only the notes from the lectures and the exercice sessions (TDs) and your personal notes are authorised. Computers, pocket calculators, and all electronic devices are forbidden.

• It is mandatory to use the same conventions as in the lectures (and as in the lecture notes), in particular, the signature of a space-time is $(-1, +1, \ldots, +1)$.

• You may write in English or French.

Good luck !

Problem 1 : Energy levels of a relativistic charged spin-0 particle in a constant magnetic field (5 points)

Consider a relativistic spin-0 particle of mass m and electric charge q in a static uniform magnetic field $\vec{B} = B\vec{e}_y$.

1-a) Write the corresponding relativistic wave-equation using a vector potential whose only non-vanishing component is A_z.

1-b) Determine the corresponding stationary solutions and their energies $E = E(n, k)$ where n is an integer and k a continuous parameter. Explicitly give the (un-normalised) wave functions and discuss the degeneracies of the spectrum. (You should use basic results from your non-relativistic quantum mechanics course, without re-deriving them, about the harmonic oscillator with potential $\frac{m}{2}\omega^2 x^2$, in particular that the eigenfunctions are given by $\varphi_n(x) \sim e^{-m\omega x^2/2} H_n(\sqrt{m\omega}x)$.)

1-c) Exhibit the non-relativistic limit of the energies (Landau levels) and explicitly give the first relativistic corrections.

1-d) Briefly discuss how the solutions would change if one uses a different vector potential $\vec{A}$.

Problem 2 : The axial current (5 points)

The classical action for a classical Dirac field ψ in the presence of a classical electromagnetic field A_μ is

$$S = \int \mathrm{d}^4x\, \overline{\psi}(x)\big(-\partial\!\!\!/ + iqA\!\!\!/(x) - m\big)\psi(x) \ , \tag{1}$$

where, as always, $\partial\!\!\!/ = \gamma^\mu\partial_\mu$, $A\!\!\!/ = \gamma^\mu A_\mu$ and $\overline{\psi} = \psi^\dagger i\gamma^0$. Consider the so-called axial transformation

$$\psi(x) \to e^{i\epsilon\gamma_5}\,\psi(x) \ , \tag{2}$$

with a constant real parameter ϵ.

2-a) Show that $e^{i\epsilon\gamma_5}\gamma^\mu = \gamma^\mu e^{-i\epsilon\gamma_5}$, and determine the corresponding transformation of $\overline{\psi}(x)$.

2-b) Show that the action S is invariant under these (simultaneous) transformations of ψ and $\overline{\psi}$ if and only if $m = 0$. For $m = 0$, determine the corresponding Noether current $j_5^\mu(x)$ (i.e. the corresponding conserved current as given by Noether's theorem). This $j_5^\mu(x)$ is called the axial current.

2-c) Write down the Dirac equations for $\psi(x)$ and for $\overline{\psi}$ (for non-vanishing mass m) and explicitly check that $\partial_\mu j_5^\mu$ is proportional to m, confirming again that the axial current is conserved if and only if $m = 0$.

Problem 3 : The current density for spin-$\frac{1}{2}$ particles (7 points)

When studying the Dirac equation in chapter 6, an important feature was that it admitted a conserved current density $j^\mu(x) = i\overline{\psi}(x)\gamma^\mu\psi(x)$, satisfying $\partial_\mu j^\mu = 0$, such that $j^0(x) \equiv \rho(x)$ is non-negative and thus could be interpreted as a probability density. However, in quantum field theory, the corresponding *normal-ordered* current operator is $J^\mu(x) = i : \overline{\Psi}(x)\gamma^\mu\Psi(x) :$, where $\Psi(x)$ is the quantum Dirac field, and $J^\mu_{\rm em} = qJ^\mu$ is the corresponding electromagnetic current density (q is the elementary charge of the particle) which is such that $J^0_{\rm em}(x)$ should take positive *and* negative values depending whether it acts on particle or anti-particle states.

Recall (or admit) that the spinors u and v satisfy the following orthonormality conditions:

$$u^\dagger(\vec p,\sigma_1)u(\vec p,\sigma_2) = v^\dagger(\vec p,\sigma_1)v(\vec p,\sigma_2) = \delta_{\sigma_1\sigma_2} \quad , \quad u^\dagger(\vec p,\sigma_1)v(-\vec p,\sigma_2) = v^\dagger(-\vec p,\sigma_1)u(\vec p,\sigma_2) = 0 \ . \tag{3}$$

3-a) Show that indeed $j^0(x) \geq 0$.

3-b) Write out $J^0(x)$ in terms of the creation and annihilation operators, and similarly for $Q = \int \mathrm{d}^3x\, J^0(x)$. Explain why this no longer is a non-negative operator.

3-c) Explicitly compute $Q\,a^\dagger(\vec p,\sigma)\,|0\rangle$, $Q\,a_c^\dagger(\vec p,\sigma)\,|0\rangle$ and $Q\,a^\dagger(\vec p_1,\sigma_1)\,a^\dagger(\vec p_2,\sigma_2)\,a_c^\dagger(\vec p_3,\sigma_3)\,|0\rangle$.

Problem 4 : Weakly relativistic limit of the Dirac equation and spin-orbit coupling (8 points)

Consider the Dirac equation for a spin-$\frac{1}{2}$ particle of electric charge q in a spherically symmetric electrostatic potential $A^0(t,\vec{x}) \equiv V(r)$, and no magnetic field so that $\vec{A} = 0$. One wants to study a weakly relativistic situation and identify the first relativistic corrections to the two-component Pauli equation. We assume that everywhere in space $|qV(r)| \ll m$.

4-a) Recall the Dirac representation of the Dirac γ-matrices and write out the Dirac equation for a stationary wave-function of positive energy $E = m + \epsilon$ written as

$$\psi(t,\vec{x}) = e^{-i(m+\epsilon)t} \begin{pmatrix} \varphi(\vec{x}) \\ \chi(\vec{x}) \end{pmatrix} . \tag{4}$$

4-b) Eliminate χ and get an exact equation for φ which one can write as

$$H_P\, \varphi(\vec{x}) = \epsilon\, \varphi(\vec{x}) , \tag{5}$$

where H_P is a differential operator of the form $H_P = f(\vec{x}) + (-i\vec{\sigma}\cdot\vec{\nabla})g(\vec{x})(-i\vec{\sigma}\cdot\vec{\nabla})$ with (possibly ϵ-dependent) functions f and g one determines. Here $(-i\vec{\sigma}\cdot\vec{\nabla}) \equiv \vec{\sigma}\cdot\vec{P}$ is meant to be a differential operator that acts on everything to its right.

4-c) Develop $g(\vec{x})$ in powers of $\frac{1}{m}$ and show that

$$H_P = qV(r) + \frac{(\vec{\sigma}\cdot\vec{P})^2}{2m} - \frac{1}{4m^2}\vec{\sigma}\cdot\vec{P}\big(\epsilon - qV(r)\big)\vec{\sigma}\cdot\vec{P} + \mathcal{O}(\frac{\epsilon^2}{m^3}) . \tag{6}$$

4-d) Use the commutator of $\vec{\sigma}\cdot\vec{P}$ and $(\epsilon - qV(r))$ and remark that, to the order we work, one can use $(\epsilon - qV(r))\varphi = \frac{P^2}{2m}\varphi$ to re-write

$$H_P = \frac{\vec{P}^2}{2m} - \frac{(\vec{P}^2)^2}{8m^3} + qV(r) + H_{\text{spin-orbit}} + \mathcal{O}(\frac{\epsilon^2}{m^3}) , \tag{7}$$

with

$$H_{\text{spin-orbit}} = \frac{a(r)}{m^2}\,\vec{S}\cdot\vec{L} + \frac{i}{m^2}\,\vec{P}\cdot\vec{r}\,b(r) , \tag{8}$$

where you will explicitly give the functions $a(r)$ and $b(r)$.

Problem 5 : Electron-neutrino scattering (15 points)

Neutrinos, just as electrons, can be described by a spin-$\frac{1}{2}$ Dirac field, but of vanishing mass. We call $\Psi_e(x)$ the quantum field of the electron and $\Psi_n(x)$ the quantum field of the neutrino. The electron (and its anti-particle, the positron) interact with the neutrino (and the anti-neutrino) via a coupling to a charged massive spin-1 particle W^+ and its anti-particle W^- with corresponding quantum field $V^\mu(x)$. The interaction Hamiltonian density is

$$\mathcal{H}_{\rm int}(x) = ig : \overline{\Psi}_e(x)\gamma^\mu(1-\gamma_5)\Psi_n(x)V_\mu^\dagger(x) : +h.c. \ , \tag{9}$$

where $+h.c.$ indicates to add the hermitian conjugate expression and g is a real coupling constant. In this exercice, we will be interested in the scattering of an electron and a neutrino.

5-a) Explicitly write out the terms $+h.c.$.

5-b) Using the notations
a_{e^-} and $a_{e^-}^\dagger$ for the annihilation and creation operators for the electron,
a_{e^+} and $a_{e^+}^\dagger$ for the annihilation and creation operators for the positron,
a_n and $a_n^\dagger$ for the annihilation and creation operators for the neutrino,
$a_{\overline{n}}$ and $a_{\overline{n}}^\dagger$ for the annihilation and creation operators for the anti-neutrino,
b_{W^+} and $b_{W^+}^\dagger$ for the annihilation and creation operators for the W^+ (particle),
b_{W^-} and $b_{W^-}^\dagger$ for the annihilation and creation operators for the W^- (anti-particle),
identify the combinations of creation and annihilation operators in $\mathcal{H}_{\rm int}$ that will be relevant to the process of the scattering of an electron and a neutrino, i.e. for a process where the initial and final states both contain one electron and one neutrino. Verify that these combinations preserve the electric charge.

5-c) Give the Feynman rules in momentum space for this theory : Write out the propagators for the electron-positron (mass m, drawn as a solid line), for the neutrino-anti-neutrino (drawn as a dotted line) and for the $W^\pm$ (mass M, use the covariant form, drawn as a wavy line). Give the interaction vertex / vertices. Give some of the factors for initial and final particles (e.g. for a final anti-neutrino and for an initial W^-).

5-d) Draw the Feynman diagram(s), that contribute(s) to the lowest non-trivial order in perturbation theory, for the scattering of an initial electron $(\vec{p}_1, \sigma_1)$ and an initial neutrino $(\vec{p}_2, \sigma_2)$ to a final electron $(\vec{p}_1{}', \sigma_1')$ and a final neutrino $(\vec{p}_2{}', \sigma_2')$.

5-e) Write out the corresponding S-matrix element. (Abbreviate $u(\vec{p}_1, \sigma_1)$ simply as $u(1)$, etc.)

5-f) Assuming from now on that $m \ll M$, show that the corresponding M-matrix element is approximately

$$M_{e:1,n:2\to e:1',n:2'} \simeq \frac{g^2}{(2\pi)^3}\,\frac{\overline{u}(2')\gamma^\mu(1-\gamma_5)u(1)\,\overline{u}(1')\gamma_\mu(1-\gamma_5)u(2)}{(p_1-p_2')^2+M^2}\ . \tag{10}$$

5-f) Indicate in a few lines (without actually doing the computation) how to obtain the corresponding unpolarised differential cross section $\frac{d\sigma}{d\Omega_{\rm cm}}$ in the center of mass frame.