ICFP Masters program
École Normale Supérieure

M1 *(Fall/Winter 2014/2015)*

Final Exam: Mathematical Aspects of Symmetries in Physics

1 Finite groups and representations

In the following, we will say a group is generated by elements $a_1, a_2, \ldots, a_n$ if all of its elements can be written as products of powers of these elements (including the trivial power $a^0 = e$).

1. Cyclic groups

   A group G is cyclic of order m if an element $a \in G$ exists such that $G = \langle a : a^m = e \rangle$, i.e. G is generated by the element a, which satisfies the given relation.

   (a) Let G be a cyclic group of order m. Suppose that $A \in GL(n, \mathbb{C})$, and define $\rho : G \to GL(n, \mathbb{C})$ by

   $$\rho : a^r \to A^r \quad (0 \leq r \leq m-1) \,. \tag{1}$$

   Show that ρ is a representation of G over $\mathbb{C}$ if and only if $A^m = 1$.

   (b) A group G is said to be simple if $G \neq \{e\}$ and the only normal subgroups of G are $\{e\}$ and G. Show that if G is a finite abelian group that is simple, then G is cyclic of prime order.

2. The symmetric group

   Define the permutations $a, b, c \in S_6$ by

   $$a = (123) \,, \quad b = (456) \,, \quad c = (23)(45) \tag{2}$$

   and let $G = \langle a, b, c \rangle$ be the subgroup of S_6 generated by these elements.

   (a) Check that

   $$a^3 = b^3 = c^2 = e \,, \quad ab = ba \,, \tag{3}$$

   $$c^{-1}ac = a^{-1} \,, \quad c^{-1}bc = b^{-1} \,. \tag{4}$$

   (b) Deduce the upper bound 18 for the order of G. What is the order of the subgroup $\langle a, b \rangle$ of G? Conclude that the order of G is 18.

   (c) Suppose that ϵ and η are complex cube roots of unity. Prove that there is a representation ρ of G over $\mathbb{C}$ such that

   $$\rho(a) = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} \,, \quad \rho(b) = \begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix} \,, \quad \rho(c) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,. \tag{5}$$

   (d) Is ρ faithful (that is 1:1) for an appropriate choice of ϵ and η? If so, for which?

3. Irreducibility

   (a) Let G be a finite group and let $\rho : G \to GL(2, \mathbb{C})$ be a representation of G. Suppose that there are elements g, h in G such that the matrices $\rho(g)$ and $\rho(h)$ do not commute. Prove that ρ is irreducible.

   (b) Assume that the representation (V, ρ) of a finite group G is reducible, $V = U \oplus W$. Consider the projection map

$$\begin{aligned} \pi : V &\to V && (6) \\ u + w &\mapsto u \quad \forall u \in U, w \in W\,. && (7) \end{aligned}$$

   Prove that the projection map is equivariant. Using this result, prove the following proposition by contradiction:

   Proposition: Let V be a representation of a finite group G, and suppose that every equivariant map from V to V is a scalar multiple of the identity map on V. Then V is irreducible.

   (c) By an application of the previous result and Schur's lemma, prove the following proposition:

   Proposition: Let $\rho : G \to GL(n, \mathbb{C})$ be a representation of the finite group G. Then ρ is irreducible if and only if every $n \times n$ matrix A which satisfies

$$\rho(g)A = A\rho(g) \quad \forall g \in G \tag{8}$$

   has the form $A = \lambda I_n$, with $\lambda \in \mathbb{C}$ and where I_n is the identity matrix on $\mathbb{C}$.

   (d) Suppose that $G = D_8 = \langle a, b : a^4 = b^2 = e, b^{-1}ab = a^{-1} \rangle$ is the group generated by the elements a and b, which satisfy the relations indicated. Check that there is a representation ρ of G over $\mathbb{C}$ such that

$$\rho(a) = \begin{pmatrix} -7 & 10 \\ -5 & 7 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} -5 & 6 \\ -4 & 5 \end{pmatrix}. \tag{9}$$

   Find all 2×2 matrices M such that $M\rho(g) = \rho(g)M$ for all $g \in G$. Hence determine whether or not ρ is irreducible, using the result from (3c).

4. Characters

   (a) The character table of S_3 is given by

	(1)	(1 2)	(1 2 3)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

   where representatives of the conjugacy classes of S_3 are listed in the first row, and χ_i denote the irreducible characters of S_3. Let χ be the class function on S_3 with the following values:

	(1)	(1 2)	(1 2 3)
χ	19	-1	-2

   Using the appropriate orthogonality relation of characters, express χ as a linear combination of χ_1, χ_2 and χ_3. Give a representation of which χ is the character.

   (b) A certain group G of order 8 is known to have a total of five conjugacy classes, with representatives $g_1, \ldots, g_5$. Four of its five irreducible characters take the following values:

g_i	$g_1 = e$	g_2	g_3	g_4	g_5
$\lvert C_G(g_i)\rvert$	8	8	4	4	4
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1

Note that beneath each representative g_i of a conjugacy class, we have given the order of the corresponding centralizer. Using the appropriate orthogonality relation of characters, find the complete character table of G. Justify each entry for χ_5.

2 Differential manifolds, Lie groups, and Lie algebras

1. Product manifolds

   Let $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ be differential manifolds of dimension d_1 and d_2 respectively. Then $M_1 \times M_2$ becomes a differential manifold of dimension $d_1 + d_2$ with differential structure $\mathcal{F}$ defined as the maximal collection containing
$$\{(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta) : (U_\alpha, \varphi_\alpha) \in \mathcal{F}_1, (V_\beta, \psi_\beta) \in \mathcal{F}_2\} \tag{10}$$
   Now consider the product manifold $M \times N$ with the canonical projections $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$.

   (a) Prove that for differential manifolds $M, N, \tilde{M}$, the map $\alpha : \tilde{M} \to M \times N$ is C^∞ if and only if $\pi_1 \circ \alpha$ and $\pi_2 \circ \alpha$ are C^∞.

   (b) Prove that the map $v \mapsto (d\pi_1(v), d\pi_2(v))$ is an isomorphism of $T_{(m,n)}M \times N$ with $T_m M \oplus T_n N$.

   (c) Let X and Y be C^∞ vector fields on M and N respectively. Then, by (b), X and Y canonically determine vector fields $\tilde{X} = (X, 0)$ and $\tilde{Y} = (0, Y)$ on $M \times N$. Prove that $[\tilde{X}, \tilde{Y}] = 0$.

   (d) Let $(m_0, n_0) \in M \times N$, and define injections $i_{n_0} : M \to M \times N$ and $i_{m_0} : N \to M \times N$ by setting
$$i_{n_0}(m) = (m, n_0), \quad i_{m_0}(n) = (m_0, n). \tag{11}$$
   Let $v \in T_{(m_0,n_0)}M \times N$, and let $v_1 = d\pi_1(v) \in T_{m_0}M$, and $v_2 = d\pi_2(v) \in T_{n_0}N$. Let $f : M \times N \to \mathbb{R}$ be C^∞. Prove that
$$v(f) = v_1(f \circ i_{n_0}) + v_2(f \circ i_{m_0}). \tag{12}$$

2. Transformations

   (a) Let M, N be differential manifolds, $\varphi : M \to N$ smooth, $d\varphi_m : T_m M \to T_{\varphi(m)}N$ for $m \in M$. Let (U_i, μ^i) and (V_i, ν^i), $i = 1, 2$, be coordinate systems on M, N respectively, $m \in U_1 \cap U_2$, $\varphi(m) \in V_1 \cap V_2$. Express $d\varphi_m$ as
$$d\varphi_m = \sum_{k,l} a^i_{kl} \frac{\partial}{\partial \nu^i_k}|_{\varphi(m)} d\mu^i_l|_m \tag{13}$$
   for $i = 1, 2$. Recall that with the notation we have introduced for differentials on functions $f : M \to \mathbb{R}$, $\{d\mu^i_l|_m\}$ for $i = 1, 2$ furnish bases for the cotangent space at the point m.
   How are the coefficients a^1_{kl} and a^2_{kl} related? We refer to this relation as the transformation property of the coefficients.

   (b) Now assume $N = \mathbb{R}$, and let $\nu^1 = \nu^2$ be the natural coordinate system on $\mathbb{R}$. How does the transformation formula simplify?

   (c) Let $\varphi_i : M \to \mathbb{R}$, $i = 1, \ldots, n$ be smooth coordinate functions on the n dimensional differential manifold M. The wedge product of the 1-forms $d\varphi_i$ is defined as
$$d\varphi_1 \wedge \ldots \wedge d\varphi_n = \frac{1}{n!} \sum_{i_1, \ldots, i_n = 1}^{n} \epsilon^{i_1 \ldots i_n} d\varphi_{i_1} \otimes \ldots \otimes d\varphi_{i_n}. \tag{14}$$
   Here, $\epsilon^{i_1 \ldots i_n}$ is the totally antisymmetric symbol defined by $\epsilon^{1 \ldots n} = 1$ and antisymmetry under transposition of two indices (thus, e.g., $\epsilon^{123} = -\epsilon^{132}$). Consider a second set of coordinate functions $\psi_i : M \to \mathbb{R}$, $i = 1, \ldots, n$. Consider $n = 2$. How are $d\varphi_1 \wedge d\varphi_2$ and $d\psi_1 \wedge d\psi_2$ related? Now consider general n. How are $d\varphi_1 \wedge \ldots \wedge d\varphi_n$ and $d\psi_1 \wedge \ldots \wedge d\psi_n$ related? What familiar object from calculus has the same transformation property?

   (d) Consider $M = \mathbb{R}^3$, μ the natural coordinate system on $\mathbb{R}^3$, and ν spherical coordinates on $\mathbb{R}^3 - \{0\}$. How are $dx_1 \wedge dx_2 \wedge dx_3$ and $dr \wedge d\varphi \wedge d\theta$ related?

3. Right invariant vector fields

Let G be a Lie group. A vector field Y on G is right invariant if Y is r_σ-related to itself for each $\sigma \in G$, where

$$r_\sigma : G \to G \tag{15}$$

$$\tau \mapsto \tau\sigma \tag{16}$$

(a) Prove that the set of right invariant vector fields on G forms a Lie algebra under the Lie bracket operation and is naturally isomorphic as a vector space with T_eG. You can assume as true that right invariant vector fields are smooth.

(b) Let $\varphi : G \to G$ be the diffeomorphism defined by $\varphi(\sigma) = \sigma^{-1}$.

i. Is φ a group homomorphism?

ii. For a vector field X on G, prove that $d\varphi \circ X \circ \varphi$ is a vector field on G.

iii. Prove that if X is a left invariant vector field on G, then $\tilde{X} = d\varphi \circ X \circ \varphi$ is a right invariant vector field on G.

iv. Prove that $\tilde{X}$ at e equals $-X(e)$. Hint: consider the differential of the map

$$G \to G \times G \to G \times G \to G \tag{17}$$

$$\sigma \mapsto (\sigma, \sigma) \mapsto (\sigma^{-1}, \sigma) \mapsto e \tag{18}$$

and use the idea behind equation (12) above.

v. Prove that $X \mapsto \tilde{X}$ gives a Lie algebra isomorphism of the Lie algebra of left invariant vector fields on G with the Lie algebra of right invariant vector fields on G.