I A variant of the REM

1. $\delta = 2$

2. $\mathcal{N} \stackrel{d}{=} \mathrm{Bin}(M,p)$, with $p = \int_{Nu}^{N(u+du)} dE \; \rho(E) = \int_{Nu}^{N(u+du)} dE \; C \; e^{-N^{x}|E|^{\delta}}$

$$= NC \int_{u}^{u+du} dv \; e^{-N^{x+\delta}|v|^{\delta}}$$

$$= NC \; e^{-N^{x+\delta}|u|^{\delta}} \; du$$

$$\mathbb{E}[\mathcal{N}] \doteq 2^N \; e^{-N^{x+\delta}|u|^{\delta}}$$

3. $x+\delta = 1$ to have the two terms of the same order , $x = 1-\delta$

in the Gaussian case one takes indeed the variance of E to be of order N

4. $\mathbb{E}[\mathcal{N}] \doteq e^{N(\ln 2 - |u|^{\delta})}$. if $|u|^{\delta} > \ln 2$ $\mathbb{E}[\mathcal{N}]$ exponentially small, with

Markov inequality $\mathbb{P}[\mathcal{N} > 0] \leq \mathbb{E}[\mathcal{N}] \longrightarrow 0$

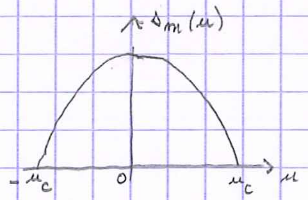
$\Rightarrow$ $\mathcal{N}_{typ} = 0$

. if $|u|^{\delta} < \ln 2$ $\mathbb{E}[\mathcal{N}]$ exponentially large , concentration via Chebychev

$$\Rightarrow \quad \mathcal{N}_{typ} \sim \mathbb{E}[\mathcal{N}] + O\left(\sqrt{\mathbb{E}[\mathcal{N}]}\right)$$

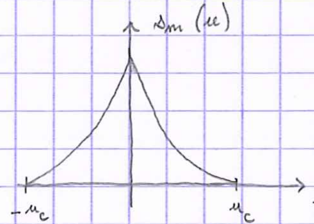
$$u_c = (\ln 2)^{1/\delta}$$

5.



$\delta > 1$

concave



$\delta < 1$

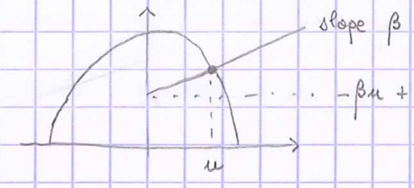
non concave , infinite derivatives in $u = 0^{\pm}$

6. $u_{gs} = -u_c$, there are states down to this energy density, but not below

7. $Z_{typ}(\beta) \doteq \int_{-u_c}^{u_c} du \; e^{N(-\beta u + s_m(u))} \doteq e^{N \sup_{u \in [-u_c, u_c]} [-\beta u + s_m(u)]}$

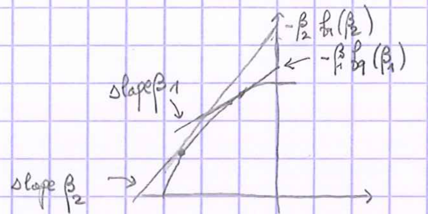
$$f_q(\beta) = -\frac{1}{N\beta} \ln Z_{typ} = -\frac{1}{\beta} \sup_{u \in [-u_c, u_c]} [-\beta u + s_m(u)]$$

8. (a)

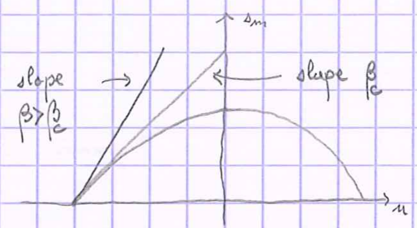


$-\beta u + s_m(u)$ is the height of the intersection between the vertical axis and the line of slope β going through $(u, s_m(u))$.

one varies . to maximize this height. for concave derivable functions, $\Leftrightarrow \beta = s_m'(u)$



(b)



for $\beta > \beta_c$ the maximiser is in $-u_c$, independently of β

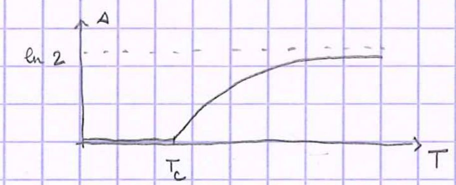
for $\beta < \beta_c$ it is in $]-u_c, 0]$

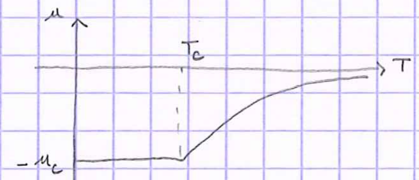
independently of $\delta > 1$ one has a condensation phase transition exactly as for $\delta = 2$

β_c such that $s_m'(-u_c) = \beta_c$, $\delta\, u_c^{\delta - 1} = \beta_c = \delta (\ln 2)^{1 - \frac{1}{\delta}}$

(c) at low T, $s = 0$, $u = -u_c$

at high T, $s \in]0, \ln 2]$, $u \in]-u_c, 0]$

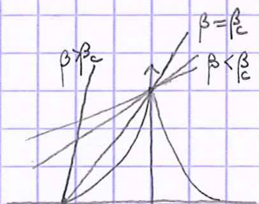




(d) the free-energy and its first derivatives (u, s) are continuous at T_c,

second derivatives discontinuous $\Rightarrow$ 2d order thermodynamically

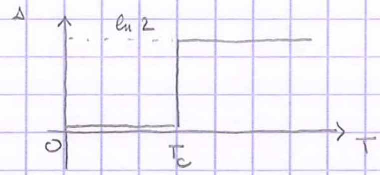
9.

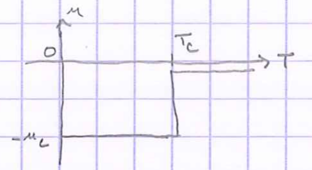


qualitatively very different: the maximizer is either in $u = 0$ (for $\beta < \beta_c$) or in $-u_c$ (for $\beta > \beta_c$)

at low T, $s = 0$, $u = -u_c$

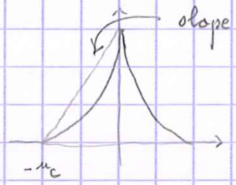
at high T, $s = \ln 2$, $u = 0$





f is continuous at T_c but not its first derivatives $\Rightarrow$ thermodynamically 1^{st} order

at β_c :

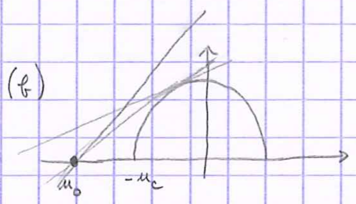


slope $\beta_c = \dfrac{\ln 2}{\mu_c} = (\ln 2)^{1-\frac{1}{\delta}}$

10. (a) $$Z = e^{-\beta N \mu_0} + \sum_{\sigma} e^{-\beta H(\sigma)}$$

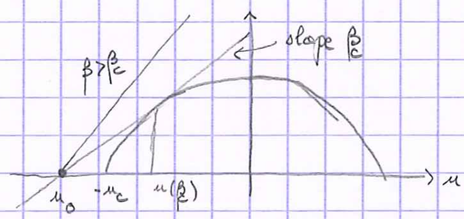
$$\Rightarrow \quad f(\beta) = -\frac{1}{\beta} \sup\left(-\beta\mu_0 \,,\, \sup_{\mu \in [-\mu_c, \mu_c]} \left[-\beta\mu + s_m(\mu)\right]\right)$$

(b)

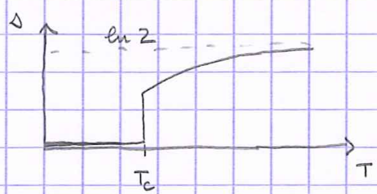


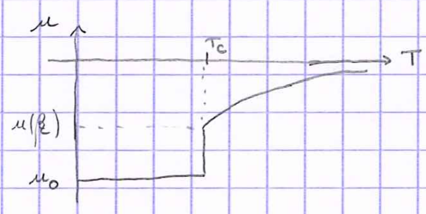
one has to compare the usual intercept with the one of a line of slope β going through $(\mu_0, 0)$

(c)



1^{st} order phase transition, μ and s are discontinuous at β_c





II Marcenko - Pastur

1. Resolvent

1. $$G(z)^{-1}_{je} = E_{je} - z\,\delta_{je} = \frac{1}{T}\xi_j^{t'}\xi_e^{t'} + \underbrace{E^{t'}_{je} - z\,\delta_{je}}_{(G^{t'}(z))^{-1}_{je}}$$

$$\sum_{ij}\xi_i^{t'}G^{t'}_{ij}(z)(G(z)^{-1})_{je} = \sum_{ij}\xi_i^{t'}G^{t'}(z)_{ij}\,(G^{t'}(z)^{-1})_{je} + \frac{1}{T}\sum_{ij}\xi_i^{t'}G^{t'}(z)_{ij}\,\xi_j^{t'}\xi_e^{t'}$$

$$= \xi_e^{t'} + \frac{1}{T}\left(\sum_{ij}\xi_i^{t'}G^{t'}(z)_{ij}\,\xi_j^{t'}\right)\xi_e^{t'}$$

2. multiplied on the right by $G(z)_{ek}$, summed over e:

$$\sum_i \xi_i^{t'}G^{t'}_{ik}(z) = \sum_e \xi_e^{t'}G(z)_{ek} + \frac{1}{T}\left(\sum_{ij}\xi_i^{t'}G^{t'}(z)_{ij}\,\xi_j^{t'}\right)\sum_e \xi_e^{t'}G(z)_{ek}$$, e renamed in i

$$\Rightarrow \sum_i \xi_i^{t'}G^{t'}_{ik}(z) = \left(1 + \frac{1}{T}\sum_{ij}\xi_i^{t'}G^{t'}(z)_{ij}\,\xi_j^{t'}\right)\sum_i \xi_i^{t'}\,G(z)_{ik}$$

multiplied on the right by $\xi_k^{t'}$, summed over k, k renamed in j, divided by T

$$\Rightarrow \frac{1}{T}\sum_{ij}\xi_i^{t'}G(z)_{ij}\,\xi_j^{t'} = \frac{\frac{1}{T}\sum_{ij}\xi_i^{t'}G^{t'}(z)_{ij}\,\xi_j^{t'}}{1 + \frac{1}{T}\sum_{ij}\xi_i^{t'}G^{t'}(z)_{ij}\,\xi_j^{t'}} \qquad (*)$$

3. $$G(z)^{-1}_{ij} = E_{ij} - z\,\delta_{ij} \quad\Rightarrow\quad \sum_{ij}(E_{ij} - z\,\delta_{ij})\,G_{ji}(z) = \sum_i \delta_{ii} = N$$

$$\Rightarrow \frac{1}{N}\sum_{ij}(E_{ij} - z\,\delta_{ij})\,G_{ji}(z) = 1$$

summing $(*)$ over t', dividing by N

$$\Rightarrow \frac{1}{N}\sum_{t'=1}^{T}\frac{\frac{1}{T}\sum_{ij}\xi_i^{t'}G^{t}_{ij}\,\xi_j^{t'}}{1 + \frac{1}{T}\sum_{ij}\xi_i G^{t'}_{ij}\,\xi_j^{t'}} = \frac{1}{N}\sum_{ij}G(z)_{ij}\underbrace{\frac{1}{T}\sum_{t'=1}^{T}\xi_i^{t'}\xi_j^{t'}}_{E_{ij} = E_{ji}}$$

$$= 1 + z\,\frac{1}{N}\,\mathrm{Tr}\,G(z) = 1 + z\,g(z)$$

4. $G^{t'}$ is independent of $\xi^{t'}$, hence $$\left\langle \frac{1}{T}\sum_{ij}\xi_i^{t'}G^{t'}_{ij}(z)\,\xi_j^{t'}\right\rangle = \frac{1}{T}\sum_{ij}\langle G^{t'}_{ij}(z)\rangle\underbrace{\langle \xi_i^{t'}\xi_j^{t'}\rangle}_{\delta_{ij}}$$

$$= q\,\langle g^{t'}(z)\rangle$$

second moment $\Rightarrow$ $$\frac{1}{T^2}\sum_{ijke}\langle \xi_i^{t'}\xi_j^{t'}\xi_k^{t'}\xi_e^{t'}\rangle\langle G^{t'}_{ij}(z)\,G^{t'}_{ke}(z)\rangle$$

$$= \frac{1}{T^2}\sigma_4\sum_i\langle (G^{t'}_{ii}(z))^2\rangle + \frac{1}{T^2}\sum_{i\neq j}\langle G^{t'}_{ij}G^{t'}_{ij} + G^{t'}_{ij}G^{t'}_{ji} + G^{t'}_{ii}G^{t'}_{jj}\rangle$$

$$= \frac{1}{T^2} (\sigma_4 - 3) \sum_i \langle (G^{t'}_{ii}(z))^2 \rangle + \frac{1}{T^2} \sum_{ij} \left[2 \langle G^{t'}_{ij}(z)\, G^{t'}_{ji}(z) \rangle + \langle G^{t'}_{ii}(z)\, G^{t'}_{jj}(z) \rangle \right]$$

$$\text{variance} = \frac{1}{T^2} (\sigma_4 - 3) \underbrace{\sum_i \langle (G^{t'}_{ii}(z))^2 \rangle}_{O(N)} + \frac{2}{T^2} \underbrace{\langle \mathrm{Tr}\left((G^{t'}(z))^2\right) \rangle}_{O(N)} + \frac{1}{T^2} \Big[\underbrace{\langle (\mathrm{Tr}\, G^{t'}(z))^2 \rangle - (\langle \mathrm{Tr}\, G^{t'}(z) \rangle)^2}_{O(N) \text{ admitting concentration}} \Big]$$

$$= O\left(\frac{1}{N}\right)$$

or consider the variance only with respect to $\xi^{t'}$, for fixed $\{\xi^t\}_{t \neq t'}$?

6. $$z g(z) = -1 + \frac{1}{q} \frac{q g(z)}{1 + q g(z)} \qquad z q g(z) = -q + \frac{1 + q g(z) - 1}{1 + q g(z)} = -q + 1 - \frac{1}{1 + q g(z)}$$

7. $$z q g(z) + z q^2 g(z)^2 = -\cancel{1} + \cancel{1} + q g(z) - q - q^2 g(z)$$

$$z q g(z)^2 + z g(z) - g(z) + q g(z) + 1 = 0$$

$$z q\, g(z)^2 + (z + q - 1)\, g(z) + 1 = 0$$

$$g(z) = \frac{-(z+q-1) \pm \sqrt{(z+q-1)^2 - 4 z q}}{2 z q}$$

3. From the resolvent to the density of eigenvalues

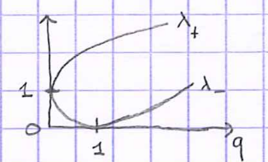
1. $$\rho(\lambda) = \frac{1}{\pi} \operatorname{Im} g(z = \lambda + i 0^+)$$

2. $\lambda_\pm$ solutions of $(\lambda + q - 1)^2 - 4 \lambda q = 0$ $\qquad \lambda^2 + 2 \lambda (q - 1 - 2q) + (q-1)^2 = 0$

$$\lambda^2 - 2\lambda (1+q) + (q-1)^2 = 0 \qquad \lambda_\pm = 1 + q \pm \sqrt{(q+1)^2 - (q-1)^2}$$

$$= 1 + q \pm 2\sqrt{q} = (1 \pm \sqrt{q})^2$$

$$\lambda_- = (1 - \sqrt{q})^2, \quad \lambda_+ = (1 + \sqrt{q})^2$$



3. $$g(z) \underset{|z| \to \infty}{\sim} -\frac{1}{z}, \qquad \sqrt{(z+q-1)^2 - 4zq} \sim |z + q - 1| \quad \text{for } z \to \pm\infty$$

$\Rightarrow$ $+$ sign for $z > \lambda_+$ to have compensation of the leading term in $g(z)$

$-$ $\qquad z < \lambda_-$

4. on $[\lambda_-, \lambda_+]$, $\quad \operatorname{Im} g(z = \lambda + i0^+) = \frac{1}{2\lambda q} \sqrt{(\lambda_+ - \lambda)(\lambda - \lambda_-)}$

there could also be a singularity in $z = 0$, but the numerator of $g(z)$

goes to $-(q-1) - |q-1|$ for $q < 1$ $\quad |q-1| = 1-q \Rightarrow$ numerator

vanishes in $z = 0$, no pole

$$\rho(\lambda) = \frac{1}{2\pi \lambda q} \sqrt{(\lambda_+ - \lambda)(\lambda - \lambda_-)} \; \mathbb{1}\left(\lambda \in [\lambda_-, \lambda_+]\right)$$

5. The support is $[\lambda_-, \lambda_+] = \left[(1-\sqrt{q})^2, (1+\sqrt{q})^2\right] \xrightarrow[q \to 0]{} \{1\}$

Indeed $q \to 0 \Leftrightarrow T \gg N$, then $E \to C$ (for instance if $T \to \infty$ with N finite)

hence $\rho_E(\lambda) \to \rho_C(\lambda) = \delta(\lambda - 1)$

6. $\xi_i^t \xi_j^t$ has rank 1, it is a projector. Thus E has rank at most T as it is a sum of T such projectors (which could be linearly dependent), as

$\operatorname{rk} E + \dim \operatorname{Ker} E = N$ one has $\underbrace{\dim \operatorname{Ker} E}_{\text{number of zero eigenvalues of } E} \geq N - T$

in $\rho(\lambda)$, $\frac{1}{N}(N-T)\,\delta(\lambda) = \left(1 - \frac{1}{q}\right)\delta(\lambda)$

7. For $q > 1$ the numerator in $g(z)$ goes to $-(q-1) - |q-1| = -2(q-1)$,

hence $g(z) \underset{z \to 0}{\sim} -\frac{2(q-1)}{2zq} = -\left(1 - \frac{1}{q}\right)\frac{1}{z}$

as $g(z) = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{\lambda_i - z}$ this pole of $g(z)$ in $z = 0$ with residue $-\left(1 - \frac{1}{q}\right)$ is precisely what was expected

8. As $T \to \infty$, for a given i,j, $E_{ij} = \frac{1}{T}\sum_{t=1}^{T} \xi_i^t \xi_j^t \longrightarrow \langle \xi_i^t \xi_j^t \rangle = \delta_{ij} = C_{ij}$ by the law of large numbers

However the density of eigenvalues depends jointly on the $N \times N$ matrix elements, this number grows, if the accuracy on $E_{ij} - C_{ij}$ does not decay fast enough with N it is possible that $\rho_E \neq \rho_C$