I Questions on the lectures

1. (a) $\int_{-\infty}^{\infty} dx\, f_X(x) = C \int_1^{\infty} dx \frac{1}{x^{5/4}} = C \left[-4 x^{-1/4} \right]_1^{\infty} = 4C \Rightarrow C = \frac{1}{4}$ by normalization of f_X

(b) $x^2 f_X(x) \underset{x\to\infty}{\sim} x^{3/4}$, $x f_X(x) \underset{x\to\infty}{\sim} x^{-1/4}$, $\frac{3}{4} > -1$, $-\frac{1}{4} > -1$

$\Rightarrow$ X admits neither a variance nor an average value

(c) $F_X(x) = \mathbb{P}[X \leq x] = \int_{-\infty}^{x} dx' f_X(x') = \int_1^x dx' \frac{1}{4}\frac{1}{x'^{5/4}} = 1 - \frac{1}{x^{1/4}}$ for $x \geq 1$

$F_{M_n}(x) = \left(F_X(x)\right)^n$ will be of order 1 when $n \to \infty$ if x is such that

$F_X(x) = 1 - O\left(\frac{1}{n}\right)$, ie for $x \sim n^4$

hence M_n is of order n^4 when $n \to \infty$

indeed $f_X(x) \sim \frac{1}{x^{1+\alpha}}$ with $\alpha = \frac{1}{4}$, from the lectures and TD one knows

that $M_n \sim n^{1/\alpha}$, with fluctuations given by a Fréchet random variable

(d) in this regime $0 < \alpha < 1$ with no average value for X we have seen that

S_n is also of order $n^{1/\alpha} = n^4$ here, with fluctuations given by a Levy stable law

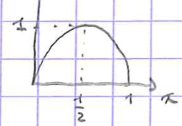
2. A quantity defined for a disordered system is said to be self averaging if, in the thermodynamic limit, it concentrates around its average value for most samples. The natural example is the free energy density, $f(\beta, \underline{J}) = -\frac{1}{N\beta} \ln Z_N(\beta, \underline{J}) \underset{N\to\infty}{\longrightarrow} \lim_{N\to\infty} -\frac{1}{N\beta} \mathbb{E}[\ln Z_N(\beta, \underline{J})]$,

the limit being called the quenched free energy

3. (a) For any value of the input, the output takes one value with probability p, the other value with probability 1-p, hence $S(Y|X) = -p \log_2 p - (1-p)\log_2(1-p) \equiv h_2(p)$

(b) $\mathbb{P}[Y=0] = \underset{X=0}{q}\ \underset{Y=X}{(1-p)} + \underset{X=1}{(1-q)}\ \underset{Y\neq X}{p}$, $S(Y) = h_2\left(q(1-p) + (1-q)p\right)$

(c) $C(p) = \sup_q \left[h_2(q(1-p) + (1-q)p) - h_2(p) \right] = \sup_q \left[h_2(q(1-p) + (1-q)p) \right] - h_2(p) = 1 - h_2(p)$

as  , hence the sup is attained for $q = \frac{1}{2}$

4. (a) There are 2 connected components in this graph

(b) There are 6 vertices, hence $\binom{6}{2} = 15$ possible edges, among which 5 are already present.

Choosing uniformly at random one of the $15 - 5 = 10$ absent edges the graph becomes connected with probability $1 - \frac{1}{10} = \frac{9}{10}$: there is only one edge whose addition does not connect the two components, the one dashed in the following figure :

5. (a) $s = \frac{1}{N} S_N \in \left\{ -1, -1 + \frac{2}{N}, \dots, 1 - \frac{2}{N}, 1 \right\}$

(b) Such a value of s is realized if $N \frac{1+s}{2}$ among the X_i are $= +1$,

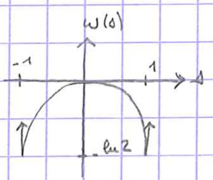
and $N \frac{1-s}{2}$ " " " $= -1$,

hence $\mathbb{P}\left[S_N = N s \right] = \binom{N}{N \frac{1+s}{2}} p^{N \frac{1+s}{2}} (1-p)^{N \frac{1-s}{2}}$

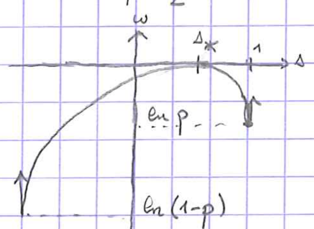
(c) Using Stirling approximation to evaluate the binomial coefficient one finds

$$w(s) = -\frac{1+s}{2} \ln\left(\frac{1+s}{2}\right) - \frac{1-s}{2} \ln\left(\frac{1-s}{2}\right) + \frac{1+s}{2} \ln p + \frac{1-s}{2} \ln(1-p)$$

(d) for $p = \frac{1}{2}$ one finds back the usual entropy function, shifted by $\ln 2$



when $p \neq \frac{1}{2}$ one adds to this function an affine function of s, hence



the maximum is in $s_* = 2p - 1 = \mathbb{E}[X] = \mathbb{E}\left[\frac{S_N}{N}\right]$,

with $w(s_*) = 0$ as this is the typical value

One can check this explicitly from the above expression of $w(s)$,

$$\frac{dw}{ds} = \frac{1}{2} \ln\left(\frac{1-s}{1+s}\right) + \frac{1}{2} \ln\left(\frac{p}{1-p}\right) = 0 \quad \text{for } \frac{1-s}{1+s} = \frac{1-p}{p}, \text{ ie } s = 2p - 1$$

then $w(2p-1) = -p \ln p - (1-p) \ln(1-p) + p \ln p + (1-p) \ln(1-p) = 0$

II A model for a polymer in a disordered environment

1. Basic properties

1. At each of the N steps one has to choose one of the k branches of the tree, there are thus k^N configurations

2. At infinite temperature the energy becomes irrelevant, the polymer will fluctuate freely and take one of the k^N configurations uniformly at random.
At zero temperature the polymer will freeze in the configuration of minimal energy, which will be unique if the disorder distribution is continuous.

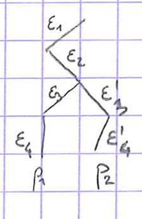
3. $$\mathbb{E}\left[E(p_1)^2\right] = \mathbb{E}\left[(\varepsilon_1+\varepsilon_3)^2\right] = \mathbb{E}\left[\varepsilon_1^2\right] + \mathbb{E}\left[\varepsilon_3^2\right] + 2\,\underbrace{\mathbb{E}\left[\varepsilon_1\varepsilon_3\right]}_{\substack{=\mathbb{E}[\varepsilon_1]\mathbb{E}[\varepsilon_3] \text{ as they are independent} \\ =0}}$$
$$= 2\sigma^2$$

$$\mathbb{E}\left[E(p_1)E(p_2)\right] = \mathbb{E}\left[(\varepsilon_1+\varepsilon_3)(\varepsilon_1+\varepsilon_4)\right] = \mathbb{E}\left[\varepsilon_1^2\right] + \underbrace{\mathbb{E}\left[\varepsilon_1\varepsilon_3\right] + \mathbb{E}\left[\varepsilon_1\varepsilon_4\right] + \mathbb{E}\left[\varepsilon_3\varepsilon_4\right]}_{0 \text{ as above}}$$
$$= \sigma^2$$

$$\mathbb{E}\left[E(p_1)E(p_3)\right] = \mathbb{E}\left[(\varepsilon_1+\varepsilon_3)(\varepsilon_2+\varepsilon_5)\right] = \mathbb{E}\left[\varepsilon_1\varepsilon_2\right] + \mathbb{E}\left[\varepsilon_1\varepsilon_5\right] + \mathbb{E}\left[\varepsilon_3\varepsilon_2\right]\,\mathbb{E}\left[\varepsilon_3\varepsilon_5\right]$$
$$= 0$$

4. In general the energies $E(p_1)$ and $E(p_2)$ of two configurations are correlated if the two paths p_1 and p_2 share some edges. More precisely, if they coincide on the first L edges, with $L \in \{0,1,\dots,N\}$ (if $L=N$ $p_1=p_2$), then

$$\mathbb{E}\left[E(p_1)E(p_2)\right] = \mathbb{E}\left[\left(\varepsilon_1+\dots+\varepsilon_L+\varepsilon_{L+1}+\dots+\varepsilon_N\right)\left(\varepsilon_1+\dots+\varepsilon_L+\varepsilon'_{L+1}+\dots+\varepsilon'_N\right)\right] = L\sigma^2$$



: an example with $N=4$, $L=2$

2. The annealed computation

1. $$\mathbb{E}\left[Z_N\right] = \mathbb{E}\left[\sum_p e^{-\beta E(p)}\right] = \sum_p \mathbb{E}\left[e^{-\beta E(p)}\right]$$ As all k^N configurations contribute in the same way, and as $E(p)$ is the sum of N independent energies ε,

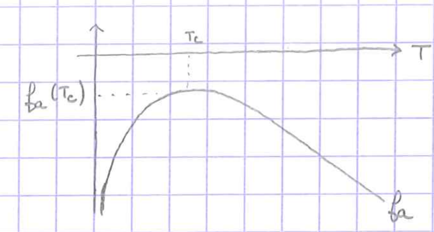
$$\mathbb{E}[Z_N] = k^N\left(\mathbb{E}\left[e^{-\beta\varepsilon}\right]\right)^N, \qquad f_a(\beta) = -\frac{1}{\beta}\ln\left(\mathbb{E}\left[e^{-\beta\varepsilon}\right]\right) - \frac{1}{\beta}\ln k$$

2. For a Gaussian distributed ε one has $\mathbb{E}\left[e^{-\beta\varepsilon}\right] = e^{\frac{1}{2}\beta^2\mathbb{E}[\varepsilon^2]} = e^{\frac{1}{2}\beta^2\sigma^2}$,

Hence $f_a(\beta) = -\beta\dfrac{\sigma^2}{2} - \dfrac{1}{\beta}\ln k$

3. $s_a(\beta) = \ln k - \beta^2\dfrac{\sigma^2}{2}$

4. $f_a(T) = -\frac{1}{T}\frac{\sigma^2}{2} - T \ln k$



one finds $T_c = \frac{\sigma}{\sqrt{2\ln k}}$,

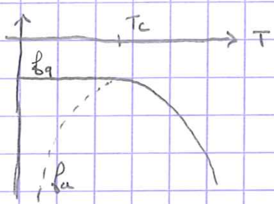
$f_a(T_c) = -\sigma\sqrt{2\ln k}$

5. Because of Jensen's inequality applied to the concave ln function, $\mathbb{E}[\ln Z_N] \leq \ln \mathbb{E}[Z_N]$

hence $f_a(\beta) \leq f_q(\beta)$

6. For $T < T_c$ the entropy computed from the annealed computation is negative, which is impossible for a model with discrete degrees of freedom, hence $f_q > f_a$ for $T < T_c$.

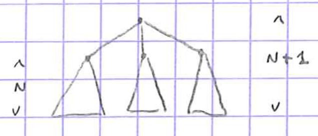
The simplest scenario, that corresponds to the one seen in the REM, is that of a freezing transition at T_c, the low temperature phase being dominated by a sub-exponential number of configurations of the polymer with the lowest possible energy density, hence a zero entropy density for $T < T_c$, corresponding to a constant f_q:



$$f_q(T) = \begin{cases} f_a(T) & \text{for } T > T_c \\ f_a(T_c) & \text{for } T < T_c \end{cases}$$

3. The quenched computation via a wave equation

1. A tree of depth $N+1$ can be viewed as k edges linked to k trees of depth N:



Each polymer configuration starts with one of the k edges, followed by a path in the corresponding subtree; its energy is the energy of the edge + the energy of the configuration in the subtree.

Hence: $$Z_{N+1} = e^{-\beta\varepsilon^{(1)}} Z_N^{(1)} + \dots + e^{-\beta\varepsilon^{(k)}} Z_N^{(k)}$$

2. $$G_{N+1}(x) = \mathbb{E}\left[e^{-e^{-\beta x} Z_{N+1}}\right] = \mathbb{E}\left[e^{-e^{-\beta x} e^{-\beta\varepsilon^{(1)}} Z_N^{(1)}} \times \dots \times e^{-e^{-\beta x} e^{-\beta\varepsilon^{(k)}} Z_N^{(k)}}\right]$$

$$= \left(\mathbb{E}\left[e^{-e^{-\beta(x+\varepsilon)} Z_N}\right]\right)^k$$ by the i.i.d character of the various branches

$$= \left(\mathbb{E}\left[G_N(x+\varepsilon)\right]\right)^k$$ by taking the average with respect to Z_N, which is independent of ε

3. The integrand is a smooth function of t, that tends to 0 faster than any power law in $t \to \pm\infty$, hence the integral is well defined. The identity is true for $z = 1$ $(0 = 0)$, the derivative of the rhs with respect to z is

$$\int_{-\infty}^{\infty} dt\, e^{-t} e^{-z e^{-t}} = \frac{1}{z}\int_0^\infty du\, e^{-u} = \frac{1}{z}$$, which is the derivative of $\ln z$,

$u = z e^{-t}$
$du = z\, dt\, e^{-t}$

hence the identity is true for all z

4. $Z_{N=1} = e^{-\beta \varepsilon^{(1)}} + \dots + e^{-\beta \varepsilon^{(k)}}$, hence the correct convention is $Z_{N=0} = 1$

which yields $G_0(x) = e^{-e^{-\beta x}}$

5. Using the identity (3) with $z = Z_N$, and changing variables with $t = \beta x$, yields

$$\ln Z_N = \beta \int_{-\infty}^{\infty} dx \left(e^{-e^{-\beta x}} - e^{-e^{-\beta x} Z_N} \right)$$

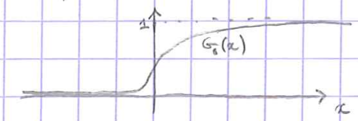
Taking the average one has by definition of $G_N(x)$ and the value of G_0:

$$\mathbb{E}\left[\ln Z_N\right] = \beta \int_{-\infty}^{\infty} dx \left(G_0(x) - G_N(x) \right)$$

4. The asymptotic solution of the wave equation

1. $G_0(x) = e^{-e^{-\beta x}}$ is an increasing function of x, tending to 0 for $x \to -\infty$, to 1 for $x \to +\infty$

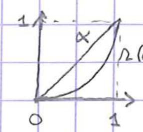
In $-\infty$ the behavior is doubly exponential, in $+\infty$ $G_0(x) \sim 1 - e^{-\beta x}$ simply exponential



2. For all realizations of Z_N the function $x \to e^{-e^{-\beta x} Z_N}$ is increasing from 0 in $x \to -\infty$ to 1 in $+\infty$, hence this is also the case for the average $G_N(x)$

3. If $G_N(x) = \alpha_N$, $\mathbb{E}\left[G_N(x+\varepsilon)\right] = \alpha_N$, hence $G_{N+1}(x) = \alpha_{N+1} = \alpha_N^k = r(\alpha_N)$

with the mapping $r(\alpha) = \alpha^k$

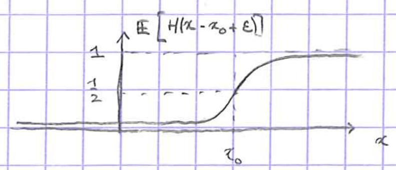


The fixed points are the solutions of $\alpha = r(\alpha)$, ie $\alpha = 0$ and $\alpha = 1$

$\alpha = 0$ is stable (a small perturbation around it is damped by r)

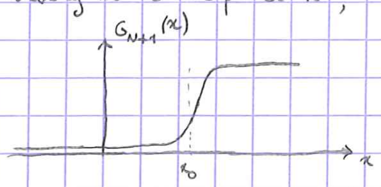
$\alpha = 1$ is unstable (——— amplified ——)

4. If $G_N(x) = H(x - x_0)$, $\mathbb{E}\left[G_N(x+\varepsilon)\right] = \int_{-\infty}^{\infty} d\varepsilon \, \rho(\varepsilon) \, \mathbb{1}(x + \varepsilon \geq x_0) = \int_{x_0 - x}^{\infty} d\varepsilon \, \rho(\varepsilon)$



generic shape for a symmetric unbounded ε, if ε is Gaussian it is exactly the "error function"

raising it to the power k,



5. The evolution equation on $G_N(x)$ has a diffusing character, because of the convolution with ε, and a non-linearity that breaks the symmetry between the stable fixed point 0 and the unstable fixed point 1, which opens the possibility to travelling wave solutions

$$g(x - (N+1)v) = \mathbb{E}\left[g(x + \varepsilon - Nv)\right]^k \underset{\substack{\text{shifting} \\ x - (N+1)v \to x}}{\iff} g(x) = \mathbb{E}\left[g(x + v + \varepsilon)\right]^k$$

Expliciting the average, $g(x) = \left(\int_{-\infty}^{\infty} d\varepsilon \; \rho(\varepsilon) \; g(x+v+\varepsilon) \right)^k$

6. One expects the stable phase ($g \simeq 0$, at $x \to -\infty$) to invade the unstable one ($g \simeq 1$, at $x \to +\infty$), hence the front separating the two phases should move towards the increasing x, ie have $v > 0$

7. $$1 - h_v(x) = \left(1 - \int_{-\infty}^{\infty} d\varepsilon \; \rho(\varepsilon) \, h_v(x+v+\varepsilon) \right)^k = 1 - k \int_{-\infty}^{\infty} d\varepsilon \; \rho(\varepsilon) \, h_v(x+v+\varepsilon) + O(h_v^2)$$

$$h_v(x) = k \int_{-\infty}^{\infty} d\varepsilon \; \rho(\varepsilon) \, h_v(x+v+\varepsilon) \quad \text{at the linear order}$$

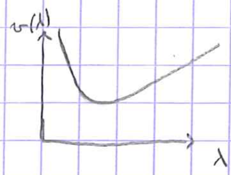
8. Inserting the exponential tail hypothesis in the equation above,

$$c \, e^{-\lambda x} = k \int_{-\infty}^{\infty} d\varepsilon \; \rho(\varepsilon) \; c \; e^{-\lambda x} \; e^{-\lambda v} \; e^{-\lambda \varepsilon}$$

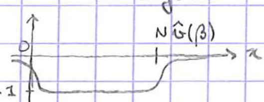
simplifying, $e^{\lambda v} = k \int_{-\infty}^{\infty} d\varepsilon \; \rho(\varepsilon) \; e^{-\lambda \varepsilon}$, hence

$$v(\lambda) = \frac{1}{\lambda} \ln k + \frac{1}{\lambda} \ln \left(\int_{-\infty}^{\infty} d\varepsilon \; \rho(\varepsilon) \, e^{-\lambda \varepsilon} \right)$$

In the Gaussian case $\int_{-\infty}^{\infty} d\varepsilon \; \rho(\varepsilon) \, e^{-\lambda\varepsilon} = e^{\frac{1}{2}\lambda^2 \sigma^2}$, $v(\lambda) = \frac{1}{\lambda} \ln k + \lambda \frac{\sigma^2}{2}$

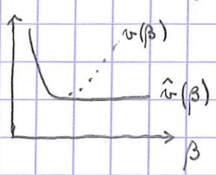


9. $$f_q(\beta) = \lim_{N \to \infty} -\frac{1}{N\beta} \mathbb{E}\left[\ln Z_N\right] = \lim_{N\to\infty} \frac{1}{N} \int_{-\infty}^{\infty} dx \; \underbrace{\left(G_N(x) - G_0(x) \right)}_{\simeq \text{ at large } N} = -\hat{v}(\beta)$$



10. One has indeed $v(\beta) = -f_a(\beta)$ by comparing the answers to the questions 2.2.1 and 2.4.8 and $G_0(x) \sim 1 - e^{-\beta x}$

11. We have seen at the question 2.4.4 that an abrupt front (corresponding to G_0 for $\beta \to \infty$) is widened by the convolution with ε, hence one cannot have a traveling wave with an arbitrary large λ and $v(\lambda)$
More generically, an initial condition containing regions with different λ will have locally different velocities, the global velocity of the stationary solution will be fixed by the slowest region.



This velocity selection principle gives

$$f_q(\beta) = \max_{\beta' \leq \beta} f_a(\beta'), \quad \text{hence in terms of temperatures}$$

$$f_q(T) = \max_{T' \geq T} f_a(T'),$$ which corresponds precisely to the freezing at T_c explained in 2.2.6

5. The replica computation

1. $Z_N^n = e^{n \ln Z_N} = 1 + n \ln Z_N + O(n^2)$ as $n \to 0$

$\mathbb{E}\left[Z_N^n\right] = 1 + n\, \mathbb{E}\left[\ln Z_N\right] + O(n^2)$, hence $\mathbb{E}\left[\ln Z_N\right] = \lim_{n \to 0} \frac{1}{n} \ln \mathbb{E}\left[Z_N^n\right]$

2. As explained in 2.1.4, $\mathbb{E}\left[E(p_a)E(p_b)\right] = N \sigma^2 q(p_a, p_b)$

3. $Z_N = \sum_p e^{-\beta E(p)}$

$Z_N^n = \sum_{p_1 \cdots p_n} e^{-\beta \sum_{a=1}^{n} E(p_a)}$ ↓ $-\beta \sum_a E(p_a)$ is a Gaussian random variable

$$\mathbb{E}\left[Z_N^n\right] = \sum_{p_1 \cdots p_n} e^{\frac{1}{2}\beta^2 \mathbb{E}\left[\left(\sum_{a=1}^{n} E(p_a)\right)^2\right]}$$

$$= \sum_{p_1 \cdots p_n} e^{N \frac{1}{2} \beta^2 \sigma^2 \sum_{a,b=1}^{n} q(p_a, p_b)}$$

$$= \int \prod_{a<b} dq_{ab}\; e^{N \frac{1}{2} \beta^2 \sigma^2 \sum_{a,b=1}^{n} q_{ab}} \underbrace{\sum_{p_1 \cdots p_n} \prod_{a<b} \delta\left(q_{a,b} - q(p_a, p_b)\right)}_{= e^{N s(Q)}}$$

where it is understood that $q_{a,b} = q_{b,a}$ and $q_{a,a} = 1$

4. $$Q = \begin{array}{c} \overset{m}{\longleftrightarrow} \\ \begin{pmatrix} \begin{smallmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{smallmatrix} & & (0) \\ & \begin{smallmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{smallmatrix} & \\ (0) & & \begin{smallmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{smallmatrix} \end{pmatrix} \\ \underset{n}{\longleftrightarrow} \end{array}$$

$\sum_{a,b} q_{ab} = \frac{n}{m} \times m^2 = n\,m$, the energetic term is thus

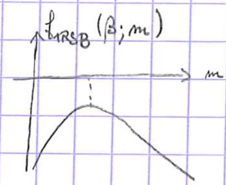
$$e^{N n\, m \frac{1}{2} \beta^2 \sigma^2}$$

One has $\frac{n}{m}$ polymer configurations to choose, one for each group, hence at the leading order $e^{N s(Q)} = \left(k^N\right)^{\frac{n}{m}} = e^{N n \frac{1}{m} \ln k}$

5. within this ansatz, $f_{1RSB}(\beta ; m) = \lim_{\substack{N \to \infty \\ n \to 0}} -\frac{1}{N\beta} \frac{1}{n} \ln \mathbb{E}\left[Z_N^n\right]$

$$= \lim_{\substack{N \to \infty \\ n \to 0}} -\frac{1}{N\beta} \frac{1}{n} \left(N n\, m \frac{1}{2} \beta^2 \sigma^2 + N n \frac{1}{m} \ln k \right)$$

$$= -\beta m \frac{1}{2} \sigma^2 - \frac{1}{\beta m} \ln k = f_a(\beta m)$$

As a function of m, 

the supremum is in $m = \frac{T}{T_c}$

For $T > T_c$ $\sup_{m \in [0,1]} f_{1RSB}(\beta ; m) = f_{1RSB}(\beta ; m = 1) = f_a(\beta) = f_q(\beta)$

For $T < T_c$ $\sup_{m \in [0,1]} f_{1RSB}(\beta ; m) = f_a(T_c) = f_q(\beta)$