I Questions on the lectures

1. $Z(\underline{J}) = \sum_{\underline{\sigma}} e^{-\beta H(\underline{\sigma})}$, $f_a = \lim_{N\to\infty} -\frac{1}{N\beta} \ln \mathbb{E}\left[Z(\underline{J})\right]$

$f_q = \lim_{N\to\infty} -\frac{1}{N\beta} \mathbb{E}\left[\ln Z(\underline{J})\right]$

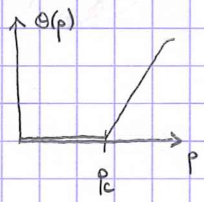
Jensen inequality, as $\ln$ concave $\mathbb{E}\left[\ln X\right] \leq \ln \mathbb{E}\left[X\right] \Rightarrow f_q \geq f_a$

2. if v is spread on N_{eff} sites, ie $v_i = \begin{cases} \frac{1}{\sqrt{N_{eff}}} & \text{on } N_{eff} \text{ sites} \\ 0 & \text{otherwise} \end{cases}$

$IPR = N_{eff} \times \frac{1}{N_{eff}^2} = \frac{1}{N_{eff}} \begin{cases} \to 0 & \text{if "delocalized"} \\ > 0 & \text{if "localized"} \end{cases}$

3. $\theta(p) =$ "density of the infinite cluster"

$= \mathbb{P}\left[\text{origin} \in \text{infinite cluster}\right]$



$\theta(p) \underset{p\to p_c^+}{\propto} (p-p_c)^\beta$

$\beta = 1$ in mean-field, as seen in the study of the Erdös-Renyi random graph

4. * X is bounded, hence admits a mean μ and a variance σ^2

by the central limit theorem, $S_n \simeq n\mu + \sqrt{n}\,\sigma\, Y$ ← $\overset{d}{=} \mathcal{N}(0,1)$

or more precisely, $\frac{S_n - n\mu}{\sqrt{n}\,\sigma} \xrightarrow{d} \mathcal{N}(0,1)$

* the maximum $M_n \to 1$ as $n\to\infty$, because 1 is the upper limit of the support of X

to be more precise one needs to study the cumulative distribution of X around 1

$F_X(x) = \int_{-\infty}^{x} dx'\, f_X(x') = \int_0^x dx'\, 2x' = x^2$

$F_X(1-\varepsilon) = (1-\varepsilon)^2 = 1 - 2\varepsilon + o(\varepsilon)$

the scale of M_n is such that $F_X = 1 - \frac{cste}{n} + o\left(\frac{1}{n}\right)$

$\Rightarrow$ here $M_n \simeq 1 - \frac{1}{n} Y$ ← r.v. of $O(1)$, with a Weibull distribution

II Hysteresis in the mean-field RFIM

2. Hysteresis loop in the thermodynamic limit

1. E depends on σ_i through : $-\frac{1}{N}\sigma_i \sum_{j\neq i}\sigma_j - \sigma_i(h_i+H) = -\sigma_i\left(\frac{1}{N}\sum_{j\neq i}\sigma_j + h_i + H\right) = -\sigma_i H_i$

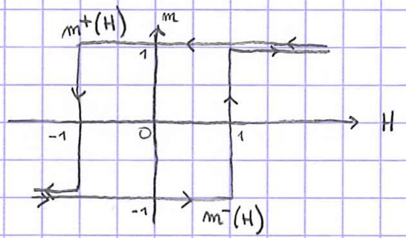
hence $\Delta E = -(-\sigma_i)H_i - (-\sigma_i H_i) = 2\sigma_i H_i(\underline{\sigma}; H, \underline{h})$

2. * A configuration is a local minima of E if $\sigma_i H_i(\underline{\sigma}; H, \underline{h}) > 0$, ie if any single spin flip would make the energy rise, hence $\sigma_i = \mathrm{sign}\left(H_i(\underline{\sigma}; H, \underline{h})\right)$ in the configurations encountered in the quasistatic protocol

* One starts with $\sigma_i = -1$ and $H_i = -\infty$. During the protocol H_i chages because :
  - H is increased
  - other spins have flipped from -1 to +1

  Both effects make H_i increase, hence σ_i will flip from -1 to +1 when H_i changes sign

3. The effective fields are $H_i = \frac{1}{N}\sum_{j=1}^{N}\sigma_j + H$ on all spins. Starting from $\underline{\sigma} = (-1,..,-1)$, $H_i = H-1$ => spins flip at $H=1$. Similarly when H decreases from $+\infty$, $H_i = H+1$ => spins flip at $H=-1$:



4. $$m(H;\underline{h}) = \frac{1}{N}\sum_{i=1}^{N}\sigma_i \underset{\underline{\sigma}\text{ local minimum}}{=} \frac{1}{N}\sum_{i=1}^{N}\mathrm{sign}\left(H_i(\underline{\sigma};H,\underline{h})\right) = \frac{1}{N}\sum_{i=1}^{N}\mathrm{sign}\left(m(H;\underline{h}) + h_i + H\right)$$

If we replace $m(H;\underline{h})$ by its limit $m(H)$,

$$m(H) = \frac{1}{N}\sum_{i=1}^{N}\mathrm{sign}\left(m(H) + h_i + H\right) \xrightarrow[N\to\infty]{} \langle \mathrm{sign}(m(H) + h + H)\rangle$$

as the h_i's are iid with distribution $\rho(h)$

5. $$m(H) = \int_{-\infty}^{\infty} dh\, \rho(h)\, \mathrm{sign}(h + m(H) + H) \qquad \mathrm{sign}(z) = -1 + 2\,\mathbb{1}(z>0)$$

$$= -1 + 2\int_{-m(H)-H}^{\infty} dh\, \rho(h)$$

$\rho(h) = \rho(-h)$

$$= -1 + 2\int_{-\infty}^{m(H)+H} dh\, \rho(h)$$

6. $R(x) = -1 + 2\int_{-\infty}^{x} dh\ \rho(h)$, affine transformation of the cumulative distribution of the h

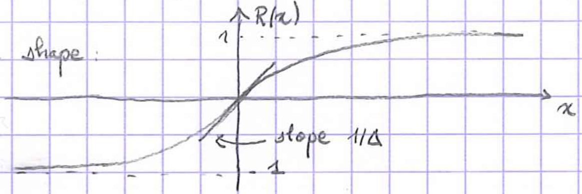
$R(-\infty) = -1$
$R(+\infty) = +1$

$$R(x) = -1 + 2\left(\int_{-\infty}^{0} dh\ \rho(h) + \int_{0}^{x} dh\ \rho(h)\right) = 2\int_{0}^{x} dh\ \rho(h) \quad \Rightarrow \quad R(-x) = -R(x)$$

$R'(x) = 2\rho(x)$, with $2\rho(x) = \frac{1}{\Delta} e^{-\frac{|x|}{\Delta}}$ $\qquad R'(0) = \frac{1}{\Delta}$

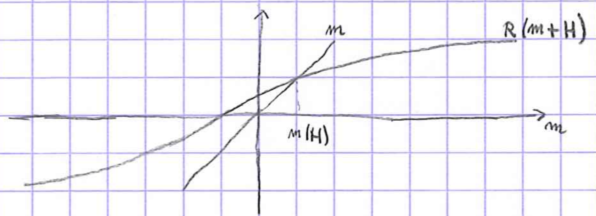
$R'(x) \nearrow$ for $x<0$, $R'(x) \searrow$ for $x>0$,

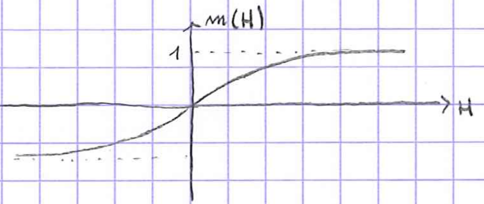
hence the shape:



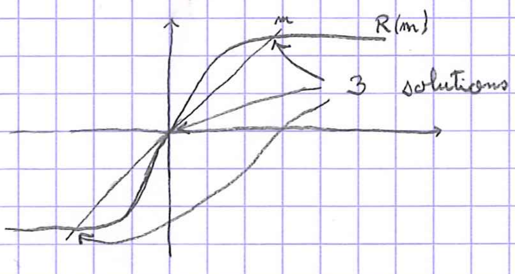
7. One plots m and $R(m+H)$, ie the translate of R, as a function of m and looks for intersections

If $\Delta > 1$ $\quad R'(0) < 1$ and $R'(x) < 1$ for all x, hence the two curves can only cross once

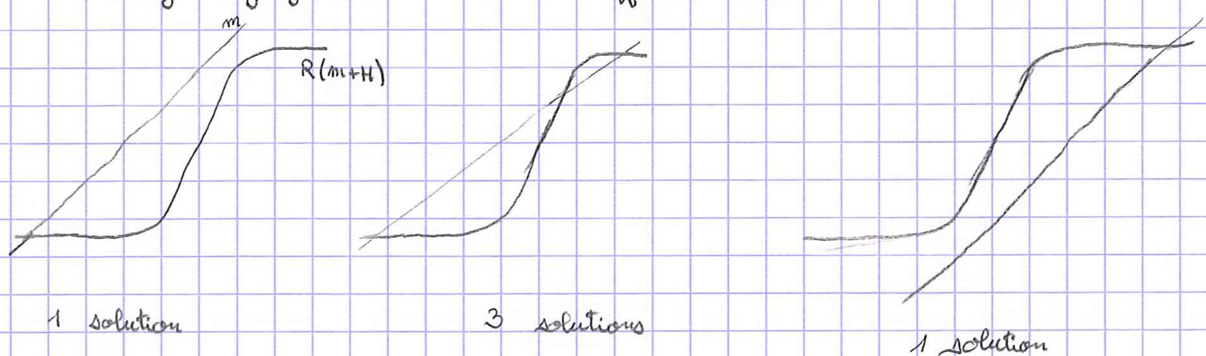




8. For instance for $H=0$, as $R'(0) > 1$



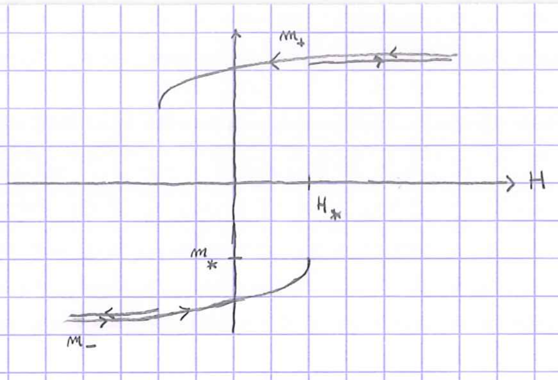
9. By changing H one has the different situations:



1 solution $\qquad$ 3 solutions $\qquad$ 1 solution

$H \ll -1$ $\qquad\qquad\qquad$ $H \gg 1$

hence

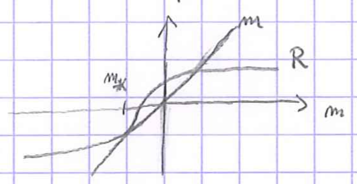


if m is solution of $m = R(m+H)$,

as $R(x) = -R(-x)$, $-m = R(-m-H)$,

which gives the symmetry $m^-(H) = -m^+(-H)$

10. at H_* the equation $m = R(m+H_*)$ has the shape :



ie $R(m+H_*)$ and m are tangent in m_*

hence $R'(m_*+H_*) = 1 = 2\rho(m_*+H_*)$

note that $m_* < 0$, hence $m_* + H_* < 0$

11. (m_*, H_*) are solutions of :

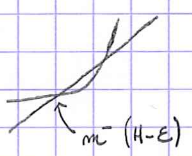
$$\begin{cases} m_* = R(m_*+H_*) \\ 1 = R'(m_*+H_*) \end{cases}$$

with $\rho(h) = \frac{1}{2\Delta} e^{-\frac{|h|}{\Delta}}$ $\quad R(x) = 2\int_0^x dh \, \frac{1}{2\Delta} e^{-\frac{|h|}{\Delta}}$

$$= \mathrm{sign}(x)\left(1 - e^{-\frac{|x|}{\Delta}}\right)$$

$$\Rightarrow \begin{cases} m_* = e^{-\frac{|m_*+H_*|}{\Delta}} - 1 \\ 1 = \frac{1}{\Delta} e^{-\frac{|m_*+H_*|}{\Delta}} \end{cases} \Rightarrow \begin{cases} m_* = \Delta - 1 \\ H_* = \Delta \ln \Delta + \Delta - 1 \end{cases}$$

12. at $H_* - \varepsilon$ the equation $m = R(m + H_* - \varepsilon)$ has the shape (zooming around m_*):



denoting $m^-(H-\varepsilon) = m_* - \delta$, one has

$m_* - \delta = R(m_* + H_* - (\varepsilon + \delta))$, where ε and δ are small.

Expanding yields $\quad m_* - \delta = \underbrace{R(m_*+H_*)}_{m_*} - (\varepsilon+\delta)\underbrace{R'(m_*+H_*)}_{1} + \frac{1}{2}(\varepsilon+\delta)^2 R''(m_*+H_*) + \cdots$

$\Rightarrow \quad \varepsilon = \frac{1}{2}\delta^2 R''(m_*+H_*)$ at lowest order, hence $\delta \propto \sqrt{\varepsilon}$

3. Avalanches and critical behavior

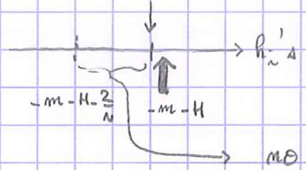
1. The effective field $H_i = m + H + h_i$ is > 0 if and only if $h_i > -m - H$

On the figure : * the spins with $H_i > 0$ (on the right of ↑) are ↑
$H_i < 0$ (on the left of ↓) are ↓ $\Big\} \Rightarrow$ stable

* if H is increased, ↑ h_5 ↓ $\quad H_5 > 0 \Rightarrow \sigma_5$ has to flip to ↑ to decrease its energy

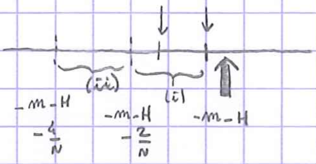
2. Once a spin flip from $\downarrow$ to $\uparrow$, $m \to m + \frac{2}{N}$, $\Uparrow$ moves further on the left

The avalanche is of size 1 if:



no other spins with h_i in this interval

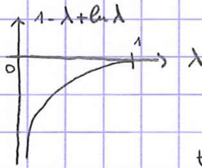
3. To have an avalanche of size 2, one needs:



- exactly one other spin in (i), to have avalanche of size ≥ 2
- no spin in (ii), otherwise the size would be > 2

4. $$P(s) \simeq \frac{e^s}{\sqrt{2\pi s}\, s^s} \frac{s^s}{s} \frac{\lambda^s}{\lambda} e^{-s\lambda} = \frac{1}{\lambda} \frac{1}{\sqrt{2\pi s^3}} e^{s(1-\lambda+\ln\lambda)}$$

5. when $H \to -\infty$, $m+H \to -\infty$, $\rho(m+H) \to 0$ $\Rightarrow$ λ starts from 0 at $H=-\infty$, and increases monotonously with H, and reaches 1 when $H = H_*$ (cf equation (2) of the text)



as long as $\lambda < 1$, $1-\lambda+\ln\lambda = -\frac{1}{s_*(H)}$ with $s_*(H) < \infty$

the tail of $P(s)$ is exponential with a characteristic size $s_*(H)$, that diverges when $H \to H_*$. At H_* the tail becomes power-law,

$P(s) \sim \text{cste} \frac{1}{s^{3/2}}$

6. divergence of a characteristic size approaching the transition, singularity of an order parameter (cf question 12 of previous part), algebraic decay right at the transition

7. $P(s) \sim \frac{1}{s^{\frac{1}{2}+1}}$ $\Rightarrow$ $\sum_{j=1}^{M} s_j \sim M^2$ from the properties of sum of heavy tailed r.v. with exponent $\alpha < 1$

The max is of the same order, the largest avalanche contributes to a finite fraction of the sum, $\delta m = \frac{2}{N} \sum_{j=1}^{M} s_j \sim \frac{1}{N} (N^{1/3})^2 \sim N^{-1/3}$

The critical avalanches are sub-extensive, when $H \gtrsim H_*$ extensive avalanches are responsible for the jump in $m^-(H)$

4. Dynamical behavior

1. with probability $1-dt$ σ_i does not attempt to flip, $\sigma_i(t+dt) = \sigma_i(t)$

   with probability dt σ_i attempts to flip, $\sigma_i(t+dt) = \text{sign}\left(\tilde{m}(t) + h_i + H_f\right)$, as it aligns with its effective field, here $\tilde{m}(t) = \frac{1}{N}\sum_{i=1}^{N} \sigma_i(t)$

   Averaging over the randomness during $[t, t+dt]$,

   $$m_i(t+dt) = (1-dt)\,\sigma_i(t) + dt \,\text{sign}\left(\tilde{m}(t) + h_i + H_f\right)$$

   Averaging over the randomness during $[0,t]$, and approximating $\tilde{m}(t)$ by $m(t)$ gives

   $$\frac{dm_i}{dt} = -m_i(t) + \text{sign}\left(m(t) + h_i + H_f\right)$$

2. In the thermodynamic limit the average over the sites amount to an average over the distribution of the h_i's, $\frac{dm}{dt} = -m(t) + \left\langle \text{sign}\left(m(t) + h + H_f\right)\right\rangle$

   At large times $\frac{dm}{dt} = 0$ $\Rightarrow$ $m(\infty) = \left\langle \text{sign}\left(m(\infty) + h + H_f\right)\right\rangle$ as in (1)

3. At large times $m(t) = m^-(H_f) - \delta(t)$, $\delta(t) \underset{t\to\infty}{\longrightarrow} 0$

   $$-\frac{d}{dt}\delta(t) = \delta(t) - m^-(H_f) + \underbrace{R\left(m^-(H_f) + H_f - \delta(t)\right)}_{\underbrace{R\left(m^-(H_f) + H_f\right)}_{m^-(H_f)} - \delta(t)\, R'\left(m^-(H_f)+H_f\right) + \frac{1}{2}\delta(t)^2 R''\left(m^-(H_f)+H_f\right) + \cdots}$$

   as long as $H_f < H^*$ $R'\left(m^-(H_f) + H_f\right) = \lambda(H_f) < 1$, $\delta(t)$ obeys

   $$-\delta'(t) = (1-\lambda)\,\delta(t) \quad \Rightarrow \quad \delta(t) \underset{t\to\infty}{\sim} \text{cst}\; e^{-(1-\lambda)t}$$, exponential decay

   with characteristic time $\frac{1}{1-\lambda}$

4. If $H_f = H^*$ then $\lambda = 1$, one has to consider the next order in the expansion,

   $$-\delta'(t) = \text{cst}\;\delta(t)^2 \quad \Rightarrow \quad \delta(t) \underset{t\to\infty}{\sim} \text{cst} \times \frac{1}{t}$$ slower power law decay