ICFP M2 - Statistical physics 2 – TD n° 9
The trap model

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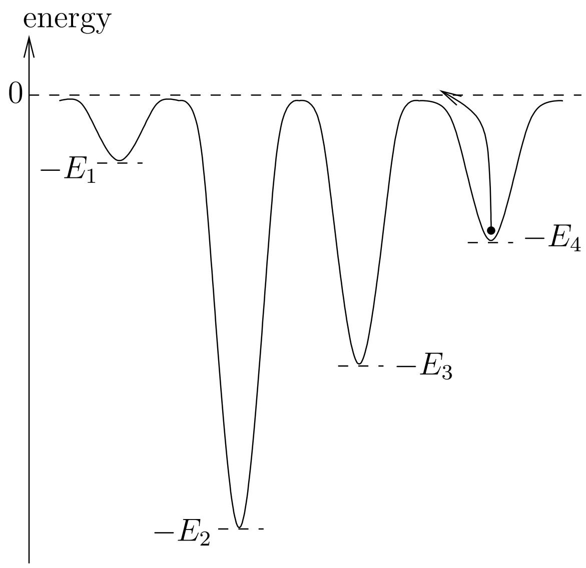
Glassy systems have a complicated energy landscape, with many local minima separated by large barriers that slows down the dynamics at low temperature. We consider in this problem a simplified dynamical model introduced by Bouchaud in 1992, which exhibits some characteristic features of more complicated systems, in particular the aging phenomenon at low temperature.

In this model the configuration space is represented as a set of N potential wells, whose depths (i.e. the opposite of their energy) are denoted $E_1, \ldots, E_N$, with $E_\alpha \geq 0$:



The state of the system is represented by a particle which, at any time t, is located at the bottom of one of the wells. The dynamics of its evolution is defined as follows: if the particle is in the well α of depth E_α it has, during an infinitesimal interval of time $\mathrm{d}t$, a probability $e^{-\beta E_\alpha}\,\mathrm{d}t$ of escaping the well, otherwise it remains in the same position. If it escapes at time t, then right afterwards it falls back into one of the N wells (including α) chosen uniformly at random.

1. Show that the probability $\widehat{P}(\alpha, t)$ of finding the particle in the α-th well at time t obeys the following master equation:

$$\frac{\partial \widehat{P}(\alpha, t)}{\partial t} = -r(E_\alpha)\widehat{P}(\alpha, t) + \left(\frac{1}{N} \sum_{\gamma=1}^{N} r(E_\gamma)\widehat{P}(\gamma, t) \right) , \qquad (1)$$

   with $r(E_\alpha) = e^{-\beta E_\alpha}$ the rate of escape per unit time from the α-th well.

2. Check that the Gibbs-Boltzmann distribution $\widehat{P}_{\mathrm{eq}}(\alpha) = \frac{1}{Z} e^{\beta E_\alpha}$ is a stationary solution of this equation.

3. Deduce that the probability $P(E,t)$ of finding the particle in a well of depth E evolves according to:

$$\frac{\partial P(E,t)}{\partial t} = -r(E)P(E,t) + \left(\frac{1}{N}\sum_{\alpha=1}^{N}\delta(E-E_\alpha)\right)\left(\int_0^\infty \mathrm{d}E'\, r(E')P(E',t)\right) \ . \tag{2}$$

4. What is the probability law of the random variable τ which gives the time spent in a well of depth E before escaping it (whether it falls back into it or not) ? Compute its average $\overline{\tau}(E)$.

We consider now that the depths E_α are i.i.d. random variables drawn from a distribution $\rho(E)$. Moreover we take the thermodynamic limit $N \to \infty$ with t finite with respect to N ; we can thus assume that the wells are visited only once, each time the particle escapes from a well we draw a new depth from $\rho(E)$, independently of everything that has happened before.

5. Show that $P(E,t)$ now obeys

$$\frac{\partial P(E,t)}{\partial t} = -r(E)P(E,t) + \rho(E)\left(\int_0^\infty \mathrm{d}E'\, r(E')P(E',t)\right) \ . \tag{3}$$

6. Show that $P_{\rm st}(E) = \frac{\rho(E)}{r(E)}$ is a (non-normalized) stationary solution of the above equation.

7. We define the two-times correlation function $C(t_{\rm w}, t_{\rm w}+t)$ as the probability that the particle has remained in the same well between the waiting time $t_{\rm w}$ and the later time $t_{\rm w}+t$. Show that

$$C(t_{\rm w}, t_{\rm w}+t) = \int_0^\infty \mathrm{d}E\, P(E,t_{\rm w})\, e^{-t\, e^{-\beta E}} \ . \tag{4}$$

8. We consider an exponential distribution for the depths of the wells, i.e. $\rho(E) = e^{-E}$ for $E \geq 0$. Show that there is a phase transition as a function of the temperature, in the sense that the stationary solution $P_{\rm st}(E)$ can be normalized only if $T > T_{\rm c}$, where you will precise the value of the critical temperature.

9. Compute the law of $\overline{\tau}$ when E is drawn from ρ. What happens at low temperature ?

10. Show that in the high temperature phase the correlation function has a stationary limit,

$$C_{\rm st}(t) = \lim_{t_{\rm w}\to\infty} C(t_{\rm w}, t_{\rm w}+t) \ , \tag{5}$$

compute this correlation function and study in particular its decay behavior at large t.

11. In the low temperature phase there is no such stationary limit, the system “ages” forever.

    (a) To understand qualitatively this aging, give the scaling of the time spent after n wells have been visited (for large n), and of the time spent in the deepest of these wells. What is the typical depth of the well in which the system is found after a large time t ?

    (b) One can show that in the low-temperature phase the solution of (3) admits, in the large time limit, a stationary solution if one changes variable from E to $u = e^{\beta E}/t$. We denote $\phi(u)$ the stationary density of this new variable, normalized as $\int_0^\infty \mathrm{d}u\, \phi(u) = 1$. Show that to have a non-trivial limit for the two-times correlation function at low temperature one needs to scale t proportionally to $t_{\rm w}$, namely to define :

$$C_{\rm ag}(\theta) = \lim_{t_{\rm w}\to\infty} C(t_{\rm w}, t_{\rm w}+\theta\, t_{\rm w}) \ . \tag{6}$$

    Express $C_{\rm ag}(\theta)$ in terms of $\phi(u)$.