# ICFP M2 - Statistical physics 2 – TD n° 7
# Wigner semi-circle law for random matrices – Solution to the last exercise

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March 9, 2020

## 3 The sketch of a proof by recursion for Wigner matrices

Here $M$ is a Wigner (real symmetric) random matrix of large size $N$.

1. We recall that the resolvent $G(z)$ is defined as $G(z) = (M - z\mathbb{I})^{-1}$, where $\operatorname{Im} z > 0$. By applying the identity (2) of the TD specified to $i = 1$ [1], we immediately obtain the formula (9) of the TD, i.e.

$$G_{11}(z) = \frac{1}{M_{11} - z - \sum\limits_{j,k=2}^{N} M_{1j}\widetilde{G}_{jk}(z)M_{k1}} \ , \tag{1}$$

where $\widetilde{G}(z)$ is the resolvent for the $(N-1)\times(N-1)$ matrix $\widetilde{M}$ obtained from $M$ by removing its first line and column.

2. By taking the average of the inverse of Eq. (1), we get

$$\mathbb{E}\left[\frac{1}{G_{11}(z)}\right] = -z - \sum_{j,k=2}^{N} \mathbb{E}[M_{1j}M_{k1}]\,\mathbb{E}[\widetilde{G}_{jk}(z)] \ , \tag{2}$$

where we have used that $\mathbb{E}[M_{11}] = 0$ together with the fact that the matrix element $\widetilde{G}_{jk}$ is independent of $M_{1j}$ and $M_{k1}$ for all $j,k \geq 2$. Furthermore, since $M$ is a Wigner matrix, which implies that $\mathbb{E}[M_{1j}M_{k1}] = \delta_{j,k}/N$, the double sum over $j$ and $k$ in (2) reduces to a single sum which can be expressed as a trace, leading to the formula (10) of the TD, i.e.

$$\mathbb{E}\left[\frac{1}{G_{11}(z)}\right] = -z - \frac{1}{N}\mathbb{E}\left[\operatorname{Tr}\widetilde{G}(z)\right] \ . \tag{3}$$

3. For $N \gg 1$, $N \simeq N-1$ and therefore $\mathbb{E}\left[\operatorname{Tr}\widetilde{G}(z)\right] \simeq \mathbb{E}\left[\operatorname{Tr} G(z)\right]$. In addition, if one assumes the concentration of $G(z)$ around $g(z)\mathbb{I}$, one has $G_{11}(z) \approx 1/g(z)$ (we recall that $g(z) = \lim_{N\to\infty} N^{-1}\mathbb{E}\left[\operatorname{Tr} G(z)\right]$), we obtain from Eq. (3) that $g(z)$ satisfies the following equation

$$\frac{1}{g(z)} = -z - g(z) \ , \qquad \text{i.e.} \qquad g^2(z) + z\,g(z) + 1 = 0 \ , \tag{4}$$

which is the same equation found in the case of the GOE in the second exercise of the TD. Hence we also find in this case that the empirical eigenvalue distribution converges to the Wigner semi-circle law $\rho_{\rm sc}(\lambda) = \sqrt{4-\lambda^2}/(2\pi)$, for $\lambda \in [-2,2]$.

[1] Although the identity (2) was shown for symmetric definite matrix, it actually holds for a wider class of invertible matrices, such as $A(z) = M - z\mathbb{I}$, thanks to the Schur's complement lemma