

ICFP M2 - STATISTICAL PHYSICS 2
Solution of the homework n° 4
Random Matrices

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The two eigenvalues of M are the solutions of its characteristic equation

$$\lambda^2 - (M_{11} + M_{22})\lambda + (M_{11}M_{22} - M_{12}^2) = 0 , \quad (1)$$

which read

$$\frac{M_{11} + M_{22}}{2} \pm \frac{1}{2} \sqrt{(M_{11} + M_{22})^2 - 4(M_{11}M_{22} - M_{12}^2)} , \quad (2)$$

hence

$$\Delta = \sqrt{(M_{11} - M_{22})^2 + 4M_{12}^2} . \quad (3)$$

Denoting $X = M_{11} - M_{22}$ and $Y = 2M_{12}$, we realize that X and Y are two independent Gaussian random variables, both of variance 2, and that $\Delta = \sqrt{X^2 + Y^2}$ can be seen as the distance from the origin of a point drawn in the plane with this distribution. Hence the density of Δ is

$$\widehat{P}(\Delta) = \int_{\mathbb{R}^2} dx dy \frac{1}{4\pi} e^{-\frac{x^2+y^2}{4}} \delta(\Delta - \sqrt{x^2 + y^2}) = \frac{1}{2} \int_0^\infty dr r e^{-\frac{r^2}{4}} \delta(\Delta - r) = \frac{1}{2} \Delta e^{-\frac{\Delta^2}{4}} , \quad (4)$$

after a change of variable towards polar coordinates. The average value of Δ is thus

$$\mathbb{E}[\Delta] = \int_0^\infty d\Delta \frac{1}{2} \Delta e^{-\frac{\Delta^2}{4}} \Delta = \sqrt{\pi} . \quad (5)$$

Changing variables from Δ to $s = \Delta/\mathbb{E}[\Delta]$ yields the probability density

$$P(s) = \widehat{P}(\Delta = s\sqrt{\pi})\sqrt{\pi} = \frac{\pi}{2} s e^{-\frac{\pi}{4}s^2} .$$