ICFP M2 - Statistical physics 2
Homework n° 4
Random Matrices

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This exercise is a preparation to the next lectures and TDs that will focus on Random Matrices and Random Hamiltonians.

Consider a two by two symmetric real random matrix M such that the matrix elements M_{11}, M_{12} and M_{22} are independent Gaussian random variables with zero mean and variances :

$$\mathbb{E}[M_{11}^2] = 1 \ , \qquad \mathbb{E}[M_{22}^2] = 1 \ , \qquad \mathbb{E}[M_{12}^2] = \frac{1}{2} \ ;$$

by symmetry $M_{21} = M_{12}$. We denote λ_1 and λ_2 the eigenvalues of M, and $\Delta = |\lambda_1 - \lambda_2|$ their spacing.

Find the probability density of Δ, its average value $\mathbb{E}[\Delta]$, and deduce that the normalized spacing $s = \Delta/\mathbb{E}[\Delta]$ has the probability density

$$P(s) = \frac{\pi}{2} s \, e^{-\frac{\pi}{4}s^2} \ ,$$

known as the Wigner surmise.

It is what Wigner proposed as an approximation for the probability density function of the normalised mean-level spacing of very complex nuclei, see Figure 1. The connection between random matrices and the Hamiltonian of a very complex non-random Hamiltonian will be discussed in the next lectures and TDs.

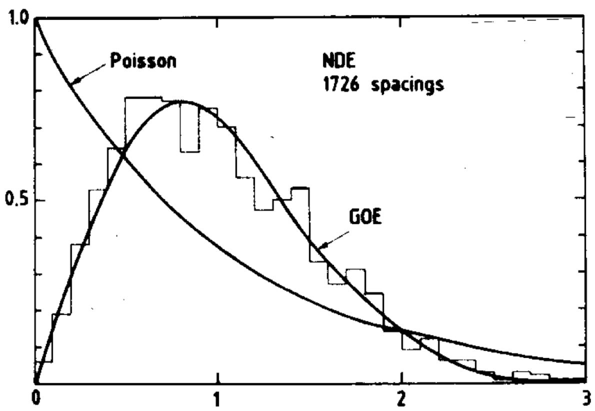

Figure 1 – Nearest neighbor spacing distribution for the "Nuclear Data Ensemble" comprising 1726 spacings (histogram) versus $s = S/D$ with D the mean level spacing and S the actual spacing. For comparison, the Wigner surmise labelled GOE is shown (don't mind about the curve labelled Poisson).