ICFP M2 - Statistical physics 2 – TD n° 4
Erdös-Rényi random graphs – Solution to the last questions

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2 Branching processes

5. Study of the critical case $c = 1$.

(a) During the TD, we obtained the equation satisfied by the generating function of S, $g(x) = \mathbb{E}(x^S)$, namely

$$g(x) = xe^{-c(1-g(x))} \, , \tag{1}$$

together with (for $c = 1$) $g(1) = 1$ and $\lim_{x \underset{<}{\to} 1} g'(x) = +\infty$.

(b) We set $x = 1 - \epsilon$ and write $g(x = 1 - \epsilon) = 1 - a\epsilon^\mu + o(\epsilon^\mu)$ as $\epsilon \to 0$. By inserting this ansatz in (1) with $c = 1$ one obtains

$$1 - a\epsilon^\mu + o(\epsilon) = (1 - \epsilon)\left(1 - a\epsilon^\mu + \frac{1}{2}a^2\epsilon^{2\mu} + o(\epsilon^{2\mu})\right) \, . \tag{2}$$

At lowest order in ϵ, one can check that the only consistent solution to (2) is

$$0 = \frac{a^2}{2}\epsilon^{2\mu} - \epsilon \Longrightarrow \begin{cases} \mu = \frac{1}{2} \, , \\ a = \sqrt{2} \, . \end{cases} \tag{3}$$

(c) One thus obtained $g(x) = 1 - \sqrt{2(1-x)}$, as $x \to 1$, which implies

$$g'(x) = \frac{1}{\sqrt{2}}(1-x)^{-1/2} + o((1-x)^{-1/2}) \, . \tag{4}$$

Recalling the definition of $g(x)$, Eq. (4) reads

$$g'(x) = \sum_{s=1}^{\infty} s\,P(s)\,x^{s-1} = \frac{1}{\sqrt{2}}(1-x)^{-1/2} + o((1-x)^{-1/2}) \, , \tag{5}$$

where $P(s) = \mathbb{P}(S = s)$. The fact that $g'(x)$ diverges as $x \to 1$ indicates that $s\,P(s)$ decays slower that $1/s$ for large s, such that the series $\sum_{s\geq 1} s\,P(s)$ is divergent. We thus assume that $P(s) \sim A/s^{\tau-1}$ for $s \to \infty$ with some amplitude A and exponent τ to be determined from (5). For this purpose, it is convenient to set $x = e^{-p}$, such that $x \to 1$ corresponds to $p \to 0$. In terms of p, the above relation (5) reads, for small p

$$e^p \sum_{s=1}^{\infty} s\,\mathbb{P}(S = s)e^{-sp} = \frac{1}{\sqrt{2}}p^{-1/2} + o(p^{-1/2}) \, . \tag{6}$$

In the small p limit, one can replace the discrete sum over s by a (Riemann) integral $e^p \sum_{s=1}^{\infty} s\,\mathbb{P}(S = s)e^{-sp} \simeq \int_1^\infty s\,\mathbb{P}(S = s)e^{-sp}ds$. By performing the change of variable $u = ps$, and substituting $P(s = u/p) \sim A(u/p)^{1-\tau}$ in the limit $p \to 0$, Eq. (7) finally leads to

$$A\,\Gamma(3-\tau)p^{\tau-3} = \frac{1}{\sqrt{2}}p^{-1/2} \, , \; p \to 0 \, , \tag{7}$$

where we have used that $\int_0^\infty u^{2-\tau} e^{-u}\, du = \Gamma(3-\tau)$. Hence Eq. (7) gives

$$\tau = \frac{5}{2}\,, \qquad A = \frac{1}{\sqrt{2\pi}}\,. \tag{8}$$

(d) The probability $\hat{P}(s)$ is the average fraction of components of an Erdös-Rényi random graph that contains exactly s vertices. On the other hand $P(s)$, that we just computed, is the average fraction of sites that belongs to a component of size s, hence one has $P(s) \propto s\hat{P}(s)$, i.e. (since it is normalised)

$$P(s) = \frac{s\hat{P}(s)}{\sum_{s'=1}^{\infty} s'\hat{P}(s')}\,. \tag{9}$$

If needed, it might be useful to convince yourself of this relation (9) on a simple example (for instance the case $N = 7$ with one component of size $S = 3$ and two of size $S = 2$). Hence, from (9), one obtains $\hat{P}(s) \propto s^{-5/2}$ for large s.

(e) To estimate the scaling (with $N \gg 1$) of the size $\mathcal{S}_N$ of the largest component at the critical point $c = 1$, let us assume that the sizes of the different connected components are i.i.d. variables, their common distribution being $\hat{P}(s)$ – this is an approximation since the sizes of the different connected components are actually correlated. In addition, we use the fact that there are $\mathcal{O}(N)$ connected components. Hence, from the results of the first lecture on extreme value statistics, since $\hat{P}(s)$ has an algebraic tail with exponent $\tau = 5/2$, one finds that $\mathcal{S}_N \sim N^{2/3}$, which coincides with the exact result.