The goal of this note is to substantiate the claim made during the lecture that the percolation transition is a critical phenomenon with the associated universality properties, and to justify the names of β and γ for the exponents describing, respectively, the growth of the probability that a given site belongs to the infinite cluster and the divergence of the average size of finite clusters, in the neighborhood of the percolation transition. To this aim we shall discuss a connection, due to Fortuyn and Kasteleyn, between percolation and the Potts model.

We consider an arbitrary finite graph $G = (V, E)$, and a Potts model defined on it. We put spin variables $\sigma_i \in \{1, \ldots, q\}$ on each vertex $i \in V$, their global configuration being denoted $\underline{\sigma} = \{\sigma_i, i \in V\}$. The spins interact ferromagnetically along the edges of the graph and with an external magnetic field in the first spin direction, in such a way that the Hamiltonian reads

$$H(\underline{\sigma}) = -J \sum_{\langle i,j \rangle \in E} \delta_{\sigma_i,\sigma_j} - h \sum_{i \in V} \delta_{\sigma_i,1} \ .$$

The partition function of this model is

$$\begin{aligned} Z &= \sum_{\underline{\sigma}} e^{-\beta H(\underline{\sigma})} = \sum_{\underline{\sigma}} \prod_{\langle i,j \rangle \in E} e^{\beta J \delta_{\sigma_i,\sigma_j}} \prod_{i \in V} e^{\beta h \delta_{\sigma_i,1}} \\ &= \sum_{\underline{\sigma}} \prod_{\langle i,j \rangle \in E} (1 + x\, \delta_{\sigma_i,\sigma_j}) \prod_i e^{\beta h \delta_{\sigma_i,1}} \qquad \text{with } \ x = e^{\beta J} - 1 \\ &= \sum_{E' \subset E} x^{|E'|} \sum_{\underline{\sigma}} \prod_{\langle i,j \rangle \in E'} \delta_{\sigma_i,\sigma_j} \prod_i e^{\beta h \delta_{\sigma_i,1}} \ . \end{aligned}$$

We have expanded here the product over the edges of G as a summation over all possible subsets E' of its edges. Let us denote $\mathcal{C}(E')$ the number of connected components of the graph (V, E'), and $n_\alpha(E')$ the number of vertices in the α-th connected component (hence α ranges from 1 to $\mathcal{C}(E')$). The summation over the spin configurations can now be performed easily : the product of Kronecker deltas enforces that the value of the spin is the same on all vertices of a connected component of (V, E'). If this value is 1 the magnetic field acts on the vertices, for the $q - 1$ other possible values it does not. This gives

$$\begin{aligned} Z &= \sum_{E' \subset E} x^{|E'|} \prod_{\alpha=1}^{\mathcal{C}(E')} \left(q - 1 + e^{\beta h n_\alpha(E')} \right) \\ &= e^{\beta J |E|} \sum_{E' \subset E} p^{|E'|} (1-p)^{|E| - |E'|} \prod_{\alpha=1}^{\mathcal{C}(E')} \left(q - 1 + e^{\beta h n_\alpha(E')} \right) \qquad \text{with } \ p = \frac{x}{1-x} = 1 - e^{-\beta J} \ . \end{aligned}$$

The parameter p is in $[0, 1]$ for ferromagnetic interactions, it can thus be interpreted as a probability. Moreover $\mathcal{P}_p(E') = p^{|E'|}(1-p)^{|E|-|E'|}$ is precisely the probability of appearance of E' in the edge percolation process on G, where each edge is kept (independently of each other) with probability p and removed with probability $1 - p$. We shall thus rewrite the last equation as :

$$Z\, e^{-\beta J |E|} = \sum_{E' \subset E} \mathcal{P}_p(E') \prod_{\alpha=1}^{\mathcal{C}(E')} \left(q - 1 + e^{\beta h n_\alpha(E')} \right) \ . \tag{1}$$

The right hand side of (1) can be viewed as a generating function, with parameters q and h, for the probability distribution of the number and size of clusters in the edge percolation process (for $h = 0$ and $q = 1$ the right hand side is equal to 1, by expanding around these values of h and q one gets the corresponding averages), while the left hand side is the partition function of the Potts model. If the latter undergoes a critical phenomenon at some critical temperature β_c, then the percolation problem will also do, at some threshold p_c.

An intuitive justification of the names β and γ of the exponents goes as follows. The spontaneous magnetization of the Potts model in the spin direction 1 can be defined as the probability that the spin at the origin of $\mathbb{Z}^d$ is in the state 1, with a boundary condition with all spins 1 on a finite portion of size $|V| = L^d$ centered around the origin, in the thermodynamic limit $|V| \to \infty$. From the connection above with the percolation model one can guess that this spontaneous magnetization will correspond to the probability of existence of a cluster of present edges in the percolation model, that connects the origin with the boundary (remember that spins take the same value in a connected component). When the boundary is sent to infinity the spontaneous magnetization of the Potts model is hence equal to the probability of existence of an infinite cluster in the percolation model, which explains the common use of the exponent β to describe the critical behavior of these two quantities. Similarly, the exponent γ describes in magnetic systems the divergence of the susceptibility $\chi = \frac{\partial m}{\partial h}$. As h is conjugated to the size of the connected component in the mapping above the magnetic susceptibility corresponds to the average size of the clusters, hence the common use of the exponent γ.

This connection can be made more quantitative, as explained now. From the expression (1) of the partition function, which makes sense for any real value of q, we obtain

$$\begin{aligned}
\frac{\partial}{\partial q} \ln Z &= \frac{\sum\limits_{E' \subset E} \mathcal{P}_p(E') \sum\limits_{\alpha=1}^{\mathcal{C}(E')} \prod\limits_{\alpha' \neq \alpha} \left(q - 1 + e^{\beta h n_{\alpha'}(E')} \right)}{\sum\limits_{E' \subset E} \mathcal{P}_p(E') \prod\limits_{\alpha=1}^{\mathcal{C}(E')} \left(q - 1 + e^{\beta h n_\alpha(E')} \right)} \\
\left. \frac{\partial}{\partial q} \ln Z \right|_{q=1} &= \frac{\sum\limits_{E' \subset E} \mathcal{P}_p(E') \sum\limits_{\alpha=1}^{\mathcal{C}(E')} e^{\beta h (|V| - n_\alpha(E'))}}{e^{\beta h |V|}} = \sum_{E' \subset E} \mathcal{P}_p(E') \sum_{\alpha=1}^{\mathcal{C}(E')} e^{-\beta h n_\alpha(E')}
\end{aligned}$$

The intensive « free-energy » can then be written as

$$\frac{1}{|V|} \frac{\partial}{\partial q} \ln Z \bigg|_{q=1} = \sum_{E' \subset E} \mathcal{P}_p(E') \frac{1}{|V|} \sum_{\alpha=1}^{\mathcal{C}(E')} e^{-\beta h n_\alpha(E')} = \sum_{E' \subset E} \mathcal{P}_p(E') \frac{1}{|V|} \sum_{x \in V} \frac{1}{n_x(E')} e^{-\beta h n_x(E')}$$

where $n_x(E')$ is the size of the connected component of vertex x in (V, E'), hence

$$\frac{1}{|V|} \frac{\partial}{\partial q} \ln Z \bigg|_{q=1} = \sum_{n=1}^{|V|} P_p(n, G) \frac{1}{n} e^{-\beta h n} \ , \tag{2}$$

with $P_p(n, G)$ defined as the average probability for a vertex to belong to a connected component of size n (averaged with respect to the reference vertex and to the percolation process).

If one takes the thermodynamic limit, calling f the limit of the left hand side of (2) and $P_p(n)$ the limit of $P_p(n, G)$ as the graph G invades $\mathbb{Z}^d$, one obtains

$$f = \sum_{n=1}^{\infty} P_p(n) \frac{1}{n} e^{-\beta h n} \ ,$$

$$\frac{1}{\beta} \frac{\partial}{\partial h} f \bigg|_{h=0} = -\sum_{n=1}^{\infty} P_p(n) = \theta(p) - 1 \tag{3}$$

$$\frac{1}{\beta^2} \frac{\partial^2}{\partial h^2} f \bigg|_{h=0} = \sum_{n=1}^{\infty} n \, P_p(n) \tag{4}$$

The first and second derivatives of the free-energy with respect to the field correspond to the spontaneous magnetization and susceptibility. In the right hand side we have introduced $\theta(p)$, probability that a vertex belongs to an infinite component (it is 1 minus the probability of belonging to a component of any finite size), hence (3) shows a quantitative connection between $\theta(p)$ and the spontaneous magnetization of the Potts model. Moreover the right hand side of (4) is the average size of a finite cluster an arbitrary vertex belongs to, which is indeed proportional to the magnetic susceptibility of the Potts model.