ICFP M2 - Statistical physics 2 – TD n° 2
The Random Energy Model - Solution of the first part

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1 Preamble : concentration of random variables

1. Denoting $\mathbb{I}(E)$ the indicator function of the event E we bound the expected value of X as
$$\mathbb{E}[X] = \mathbb{E}[X\,\mathbb{I}(X \geq a)] + \mathbb{E}[X\,\mathbb{I}(X < a)] \geq \mathbb{E}[X\,\mathbb{I}(X \geq a)] \geq a\,\mathbb{E}[\mathbb{I}(X \geq a)] = a\,\mathbb{P}[X \geq a] \ ,$$
where the first inequality holds because X is a positive random variable. The Markov inequality follows by dividing by a.

2. If we apply the Markov inequality to the random variable $Y = (X - \mathbb{E}[X])^2$, which is clearly positive, we get
$$\mathbb{P}[Y \geq a] \leq \frac{1}{a}\mathbb{E}[Y] = \frac{1}{a}\mathrm{Var}[X] \ . \quad (1)$$
We can then obtain the Chebychev inequality as
$$\mathbb{P}[|X - \mathbb{E}[X]| \geq t\,\sqrt{\mathrm{Var}[X]}] = \mathbb{P}[(X - \mathbb{E}[X])^2 \geq t^2\,\mathrm{Var}[X]] \leq \frac{1}{t^2} \ , \quad (2)$$
applying (1) with $a = t^2\,\mathrm{Var}[X]$.

3. Note that for a random variable X taking values in 0,1, …, one has $X > 0$ if and only if $X \geq 1$. Hence the Markov inequality with $a = 1$ immediately gives
$$\mathbb{P}[X > 0] \leq \mathbb{E}[X] \ .$$
From Chebychev inequality with $t = \frac{\mathbb{E}[X]}{\sqrt{\mathrm{Var}[X]}}$ we obtain, for any random variable admitting a variance,
$$\mathbb{P}[|X - \mathbb{E}[X]| \geq \mathbb{E}[X]] \leq \frac{\mathrm{Var}[X]}{\mathbb{E}[X]^2} \ . \quad (3)$$
As $|X - \mathbb{E}[X]| \geq \mathbb{E}[X] \Leftrightarrow X \leq 0 \ \text{ or } \ X \geq 2\,\mathbb{E}[X]$, this can be rewritten
$$\mathbb{P}[X \leq 0] + \mathbb{P}[X \geq 2\,\mathbb{E}[X]] \leq \frac{\mathrm{Var}[X]}{\mathbb{E}[X]^2} \ . \quad (4)$$
For the non-negative integer valued random variable considered here $X \leq 0$ if and only if $X = 0$, and the probability of $X \geq 2\,\mathbb{E}[X]$ is a non-negative number, hence
$$\mathbb{P}[X = 0] \leq \frac{\mathrm{Var}[X]}{\mathbb{E}[X]^2} \ , \qquad \text{or equivalently} \qquad \mathbb{P}[X > 0] \geq 1 - \frac{\mathrm{Var}[X]}{\mathbb{E}[X]^2} \ . \quad (5)$$
To obtain an improved bound we shall use the Cauchy-Schwarz inequality, which states that for two random variables A and B one has $\mathbb{E}[AB] \leq \sqrt{\mathbb{E}[A^2]}\sqrt{\mathbb{E}[B^2]}$. Applying this to $A = X$, $B = \mathbb{I}(X > 0)$ yields, for these non-negative integer valued random variables X,
$$\mathbb{E}[X] = \mathbb{E}[X\mathbb{I}(X > 0)] \leq \sqrt{\mathbb{E}[X^2]}\sqrt{\mathbb{E}[\mathbb{I}(X > 0)^2]} = \sqrt{\mathbb{E}[X^2]}\sqrt{\mathbb{P}[X > 0]} \ ; \quad (6)$$
in the last step we used the fact that the square of an indicator function is equal to itself. Squaring this inequality and dividing it by $\mathbb{E}[X^2]$ gives finally
$$\frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]} \leq \mathbb{P}[X > 0] \ , \qquad \text{i.e.} \qquad \mathbb{P}[X > 0] \geq 1 - \frac{\mathrm{Var}[X]}{\mathbb{E}[X^2]} \ , \quad (7)$$
which is stronger than the inequality (5) obtained from Chebychev as $\mathbb{E}[X^2] \geq \mathbb{E}[X]^2$.