

ICFP M2 - STATISTICAL PHYSICS 2 – TD n° 3

The mean-field p -spin glass model

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In this TD we shall study with the replica method the thermodynamics of the fully connected p -spin glass model, defined by its Hamiltonian

$$H(\underline{\sigma}; \underline{J}) = - \sum_{1 \leq i_1 < i_2 < \dots < i_p \leq N} J_{i_1 i_2 \dots i_p} \sigma_{i_1} \dots \sigma_{i_p} . \quad (1)$$

The Ising spins σ_i have p -body interactions ($p = 2$ corresponds to the Sherrington-Kirkpatrick model), the coupling constants $J_{i_1 \dots i_p}$ are Gaussian i.i.d. random variables of zero mean and variance $\frac{p!}{2N^{p-1}}$. We denote $\mathbb{E}[\bullet]$ the average over these random couplings.

We will be mostly interested here in the case $p \geq 3$; even though these multi-body interactions do not seem microscopically motivated, the properties of this model has strong similarities with the ones of the structural glasses, and a mean-field theory for the glasses, called Random First Order Transition, was built starting from the p -spin model. Moreover this type of interaction appears naturally in the interdisciplinary applications to computer science.

1. Show that the energies $H(\underline{\sigma}; \underline{J})$ are correlated Gaussian random variables with zero mean and covariance

$$\mathbb{E}[H(\underline{\sigma}; \underline{J})H(\underline{\tau}; \underline{J})] = N \frac{1}{2} q(\underline{\sigma}, \underline{\tau})^p (1 + o(1)) \quad (2)$$

when $N \rightarrow \infty$, where $q(\underline{\sigma}, \underline{\tau}) = \frac{1}{N} \sum_{i=1}^N \sigma_i \tau_i$ is the overlap between the two configurations.

2. Explain why this model should become equivalent to the random energy model in the limit $p \rightarrow \infty$ (taken after the thermodynamic limit $N \rightarrow \infty$).
3. Compute the annealed free-energy $f_a(\beta)$ of the p -spin model.

The computation made during the lectures showed that, when n is a positive integer,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \ln \mathbb{E}[Z(\beta, \underline{J})^n] = \sup_Q A(Q) , \quad \text{with} \quad A(Q) = n \frac{\beta^2}{4} + \frac{\beta^2}{4} \sum_{a \neq b} q_{ab}^p + S(Q) , \quad (3)$$

where $Q = \{q_{ab}\}$ is an $n \times n$ matrix, with 1 on the diagonal, encoding the overlaps between the n replicas of the system and $S(Q)$ the entropy of such configurations. The latter term can be computed to obtain

$$A(Q) = n \frac{\beta^2}{4} + n \ln 2 - \frac{\beta^2}{4} (p-1) \sum_{a \neq b} q_{ab}^p + \ln \left(\frac{1}{2^n} \sum_{\sigma^1, \dots, \sigma^n} \exp \left[\frac{\beta^2}{4} p \sum_{a \neq b} q_{ab}^{p-1} \sigma^a \sigma^b \right] \right) . \quad (4)$$

To determine the quenched free-energy we want to use the replica trick and express

$$f_q(\beta) = -\frac{1}{\beta} \lim_{n \rightarrow 0} \frac{1}{n} A(Q_*) , \quad (5)$$

where Q_* is the saddle-point dominating A . To take the limit $n \rightarrow 0$ we have to make an ansatz on the form of Q , as we shall now discuss.

4. We start with the simplest and most natural Replica Symmetric (RS) form of the matrix Q , with $q_{ab} = q \geq 0$ for all $a \neq b$.

(a) Show that such a saddle-point yields the following free-energy,

$$f_{\text{RS}}(q; \beta) = -\frac{\beta}{4} - \frac{1}{\beta} \ln 2 + \frac{\beta}{4}(pq^{p-1} - (p-1)q^p) - \frac{1}{\beta} \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \ln \cosh \left(\beta \sqrt{\frac{pq^{p-1}}{2}} z \right) ;$$

to perform this computation you should use the identity

$$\int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2 + az} = e^{\frac{1}{2}a^2} . \quad (6)$$

(b) Check that $f_{\text{RS}}(q=0; \beta) = f_{\text{a}}(\beta)$, the annealed free-energy.

(c) Analyze the behavior of the various terms of f_{RS} in the limit $q \rightarrow 0$, and conclude that $q=0$ is a local maximum for $p \geq 3$.

(d) The best estimate of the quenched free-energy within the RS ansatz is $f_{\text{RS}}(\beta) = \sup_{q \in [0,1]} f_{\text{RS}}(q; \beta)$

(the maximization instead of the usual minimization being a counter-intuitive consequence of the $n \rightarrow 0$ limit). Assuming that $q=0$ is the global maximum, argue that a phase transition must occur for some $\beta \leq 2\sqrt{\ln 2}$ (you should compute the entropy associated to f_{RS}).

5. This phase transition manifests itself as Replica Symmetry Breaking (RSB) of the relevant saddle-point Q_* . The simplest way to break the symmetry between the n replicas is to divide them into n/m groups of m replicas, and to take $q_{ab} = q_1$ for the off-diagonal elements of the n/m diagonal blocks of the matrix Q , i.e. when $a \neq b$ are two distinct replicas of the same group, and $q_{ab} = q_0$ in the off-diagonal blocks, i.e. when a and b are in different groups. This is the first level of replica symmetry breaking (1RSB). In the present model one can actually take $q_0 = 0$, as suggested by the RS solution.

(a) Compute $A(Q)$ for such a matrix, and take the limit $n \rightarrow 0$ with $m \in [0, 1]$ to obtain

$$\begin{aligned} f_{\text{1RSB}}(q_1, m; \beta) &= -\frac{\beta}{4} - \frac{1}{\beta} \ln 2 - (1-m) \frac{\beta}{4} (p-1) q_1^p \\ &+ \frac{\beta}{4} p q_1^{p-1} - \frac{1}{\beta m} \ln \left[\int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \cosh^m \left(\beta \sqrt{\frac{p q_1^{p-1}}{2}} z \right) \right] . \end{aligned}$$

(b) How does this expression simplify when $q_1 = 0$? and when $m = 1$?

The estimate of the quenched free-energy of the model at the 1RSB level is obtained by maximizing $f_{\text{1RSB}}(q_1, m; \beta)$ with respect to q_1 and m , both in the interval $[0, 1]$. The equations obtained by imposing the stationarity of f_{1RSB} with respect to q_1 and m have different type of solutions depending on the temperature :

- At high temperature, above a temperature that we call T_d , there is only one solution corresponding to $q_1 = 0$. In this regime we recover the high temperature (replica symmetric) solution discussed previously. The temperature T_d depends on p : for $p = 3$ one finds $T_d \simeq 0.681598$, whereas for $p \rightarrow \infty$ one has $T_d \rightarrow \infty$.
- Below T_d and above a temperature that we call T_c one finds two solutions: (1) the high temperature solution discussed before, (2) a new one corresponding to $m = 1$ and $q_1 > 0$. As it can be easily checked, they have the same free energy. The temperature T_c depends on p : for $p = 3$ one finds $T_c \simeq 0.651385$, whereas for $p \rightarrow \infty$ one finds $T_c \rightarrow \frac{1}{2\sqrt{\ln 2}}$, which is the critical temperature of the random energy model.
- Below T_c the optimal solution corresponds to $0 < m < 1$ and $q_1 > 0$. The high temperature solution exists at any temperature, as discussed previously, but is not optimal for $T < T_c$.

The interpretation of these transitions will be discussed in the next lecture.