ICFP M2 - Statistical physics 2
Homework n^o 1

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In the TD 1 we have studied the distribution of the maximum M_n of a large number n of independent and identically distributed random variables $X_1, \dots, X_n$. One can investigate more detailed extremal properties of such large samples of random variables, for instance :

— what is the law of the second largest variable among $X_1, \dots, X_n$?

— what is the law of the k-th largest variable among $X_1, \dots, X_n$, for arbitrary k ?

— what is the joint law of the k largest elements in $X_1, \dots, X_n$? (you can convince yourself that they are indeed correlated)

— is it possible to answer this last question in the limit $k \to \infty$ (after $n \to \infty$) ?

To answer some of these questions we suggest the following approach :

— from the independent random variables $X_1, \dots, X_n$ define $\widehat{X}_1, \dots, \widehat{X}_n$ with $\widehat{X}_i = (X_i - a_n)/b_n$, where a_n and b_n are the series introduced in the TD that define the rescaling under which $(M_n - a_n)/b_n$ has a non-trivial limit.

— call $N_n([u, v])$ the (random) number of points $\widehat{X}_i$ among $\widehat{X}_1, \dots, \widehat{X}_n$ which falls in the interval $[u, v]$.

— determine the probability distribution of $N_n([u, v])$, and of its limit $N([u, v])$ as $n \to \infty$.

— characterize the joint law of $N_n([u_1, v_1]), \dots, N_n([u_p, v_p])$ when the intervals $[u_i, v_i]$ are disjoint, and then take the limit $n \to \infty$.

— find back from this approach the distribution of the maximum derived in the TD, and generalize this result to the k-th maximum.

— conditionally on the event $N([u, v]) = p$, describe the joint law for the p points $\widehat{X}_i$ that fall in the interval $[u, v]$.