Quantum field theory: pre-course notes

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1 Groups and representations

Quantum field theory was born out of the marriage of quantum mechanics and special relativity. This marriage requires that the Lorentz group act on the Hilbert space $\mathcal{H}$ underlying the quantum theory. $\mathcal{H}$ being a linear space, the action of the Lorentz group on it must be linear. Let us quickly review some of the mathematics underlying these notions.

Definition 1. *A group $G = (S, \cdot)$ is a set S together with an operation*

$$\cdot \quad : \quad S \times S \to S \tag{1.1}$$

such that

- $\exists e \in S : \quad e \cdot a = a \cdot e = a \quad \forall a \in S$ *(existence of identity)*
- $\forall a \in S \quad \exists a^{-1} \in S : \quad a \cdot a^{-1} = a^{-1} \cdot a = e$ *(existence of inverse)*
- $\forall a, b, c \in S : \quad (a \cdot b) \cdot c = a \cdot (b \cdot c)$ *(associativity)*

You can amuse yourself by proving some simple consequences of these axioms, such as uniqueness of the identity, uniqueness of inverses, etc.

Examples of groups are

- permutation group on n letters (finite) ,
- rotation group in d dimensions (infinite, compact),
- Lorentz group in d dimensions (infinite, non-compact).

Definition 2. *Given a vector space V, a representation (ϕ, V) of a group G is a homomorphism*[1]

$$\phi: \quad G \to \mathrm{Aut}(V)\,. \tag{1.2}$$

Recall that the automorphisms of a vector space are the invertible linear maps between the space and itself. Convince yourself that the space of automorphisms furnishes a group, with composition as the group operation. In the case that V is finite dimensional, this group is also called $GL(V)$. Upon choosing a basis for V, it is isomorphic (non-canonically) to the group of invertible $d \times d$ matrices, where d denotes the dimension of V.

Definition 2 encapsulates what we mean when we say that a group acts on a vector space: each element $g \in G$ of the group maps to a map $\phi(g) : V \to V$; $\phi(g)$ acts on V.

In these terms, we require that the Hilbert space furnish a representation of the Lorentz group. Throughout this semester, we will discover that a lot of physics follows from this seemingly innocuous requirement.

Note: In physics, we often encounter groups as sets of transformations on a vector space, with the group operation given by composition. E.g. we define the rotation group as the set of matrices enacting rotations on $\mathbb{R}^3$, together with composition. To use the language of representations, we would first introduce an abstract group G whose elements are in 1:1 correspondence with rotation matrices, and whose multiplication is induced from composition of rotation matrices. Having thus obtained G, the rotation matrices furnish a representation of G on the vector space $\mathbb{R}^3$. Put this way, the distinction between G and the set of rotation matrices might seem vacuous. It is not: G possesses other than 3 dimensional representations which are equally interesting. In fact, we will see that different representations of the rotation group are intimately related to the physical notion of spin.

2 The Lorentz group – preliminaries

The main group of interest in this course will be the Lorentz group. We will study various aspects of this group in more detail in the course, but we will need some preliminary notions to get off the ground.

We introduce the Minkowski metric

$$(\eta_{\mu\nu}) = \mathrm{diag}(1, -1, -1, -1)\,, \quad \mu, \nu = 0, \dots 3\,. \tag{2.1}$$

The notation (A_{ab}) indicates that we are considering the matrix A as a whole (and not merely an entry A_{ab}), but we also wish to indicate where we will place the indices should we require to indicate matrix entries (we thus distinguish between A_{ab} and A^{ab}, see below).

[1] A homomorphism ϕ between two spaces M and N is a map that respects the structure on these spaces. If M and N are linear spaces, e.g., then a homomorphism is a linear map. If M and N are groups, the case of interest for us, then $\phi(g_1 \cdot g_2) = \phi(g_1) \cdot \phi(g_2)\ \forall g_1, g_2 \in M$. Note that we are using the same symbol $\cdot$ to denote the operation on M and N, and that we are no longer distinguishing between the group and the underlying set (called G and S above).

Note that we could have equally well introduced the metric with opposite sign convention. The convention we have chosen is handy because e.g. $p^2 = m^2$, but has the disadvantage that it restricts to *minus* the Euclidean norm on space.

We introduce the Lorentz group $O(1,3)$ via its action on $\mathbb{R}^4$ (representing e.g. coordinates on space-time); compare the note at the end of the previous section.

Definition 3.

$$O(1,3) := \{\Lambda \in GL(4,\mathbb{R}) \,|\, \eta(\Lambda\cdot,\Lambda\cdot) = \eta(\cdot,\cdot)\} \tag{2.2}$$

O stands for orthogonal, $(1,3)$ indicates the signature of the metric. The analogy to the rotation group is apparent. We will discuss the connected components of the Lorentz group in class at a later point.

For computations, it is often useful to write out relations in components. E.g. for $a, b \in \mathbb{R}^4$,

$$\eta(a,b) = \eta_{\mu\nu} a^\mu b^\nu = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3\,, \quad (\Lambda a)^\mu = \Lambda^\mu{}_\nu a^\nu\,, \tag{2.3}$$

and hence

$$\eta(\Lambda a, \Lambda b) = \eta_{\mu\nu} \Lambda^\mu{}_\rho a^\rho \Lambda^\nu{}_\sigma a^\sigma\,. \tag{2.4}$$

We thus obtain

$$\Lambda \in O(1,3) \quad \Leftrightarrow \quad \eta_{\rho\sigma} \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu = \eta_{\mu\nu}\,. \tag{2.5}$$

Here are a few conventions regarding index placement and ensuing consequences:

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  $$\eta^{\mu\nu} := (\eta^{-1})_{\mu\nu} \quad \Rightarrow \quad \eta^{\mu\nu}\eta_{\nu\rho} = \delta^\mu{}_\rho\,, \tag{2.6}$$

  where we have introduced the notation $(\delta^\mu{}_\nu) = \mathrm{diag}(1,1,1,1)$.

- If we refer to the space $\mathbb{R}^4$ as V when it carries the above representation of the Lorentz group, we can introduce higher dimensional representations of the Lorentz group by considering tensor products $V^{\otimes k} = V \underbrace{\otimes \ldots \otimes}_{k \text{ times}} V$. Such representations are called tensor representations, and elements of the underlying vector space are called rank k tensors. Expanding a tensor A in the canonical basis on $V^{\otimes k}$ induced by a basis of V, we obtain the components of A, which carry k indices.

- We raise and lower indices with $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$. You can work out as an exercise what this mapping is in terms of representation theory.[2] Hence,

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    $$\eta_{\mu\nu} a^\mu b^\nu = a_\mu b^\mu = a^\mu b_\mu \tag{2.7}$$

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    $$\eta_{\mu\nu} \eta^{\nu\rho} = \eta_\mu{}^\rho = \delta_\mu{}^\rho \tag{2.8}$$

[2] To get you headed in the right direction, note that $\eta(v,\cdot)$, $v \in V$, is an element of the dual space $V^\vee$. The representation on V induces a representation on $V^\vee$.

- We can express the entries of the inverse Λ^{-1} of a Lorentz transformation Λ in terms of the entries of Λ. By definition of the inverse,

$$\delta^\mu{}_\nu = (\Lambda^{-1} \circ \Lambda)^\mu{}_\nu = (\Lambda^{-1})^\mu{}_\rho \Lambda^\rho{}_\nu \,. \tag{2.9}$$

  By the defining property of a Lorentz transformation,

$$\eta_{\mu\nu} = \eta_{\sigma\rho} \Lambda^\sigma{}_\mu \Lambda^\rho{}_\nu \tag{2.10}$$

  Raising the μ index, we obtain

$$\eta^\mu{}_\nu = \eta_{\sigma\rho} \Lambda^{\sigma\mu} \Lambda^\rho{}_\nu = \Lambda_\rho{}^\mu \Lambda^\rho{}_\nu \,. \tag{2.11}$$

  Acting on the right by Λ^{-1}, we conclude

$$(\Lambda^{-1})^\mu{}_\rho = \Lambda_\rho{}^\mu \,. \tag{2.12}$$

3 Classical field theory

Classical field theory is a rich subject in its own right. We will unfortunately have very little time to do it justice in this course.

Essentially, we will transition from discrete to continuous systems via the replacements

$$\begin{array}{ccc} \phi_i \,, \pi_i & \longrightarrow & \phi(x), \pi(x) \\ \{\phi_i, \pi_j\} = \delta_{i,j} & \longrightarrow & \{\phi(x), \pi(y)\} = \delta(x-y) \\ \sum_i & \longrightarrow & \int dx \end{array}$$

We are thus transitioning from a finite index set $i \in I$ to a continuous index set $x \in \mathbb{R}$ (or $\mathbb{R}^2, \mathbb{R}^3, \ldots$). Unlike finite sums, integrals do not always exist. To ensure that they do, one needs to carefully specify the function spaces involved. We will have little occasion to do so.

Do not be confused by the fact that canonical variables in analytical mechanics are often called x_i, p_i, whereas in the continuous setting, one often chooses to label elements of the continuous indexing set by the letter x.

Many manipulations in analytical mechanics require differentiating with regard to the canonical variables (e.g. in order to derive the equations of motion). One can introduce a formal device called a functional derivative to mimic this operation at the level of fields. If we agree to only work with Hamiltonians H and Lagrangians L obtained from corresponding densities $\mathcal{H}(x)$ and $\mathcal{L}(x)$,[3]

$$H = \int d^3x \, \mathcal{H}(x) \,, \quad L = \int d^3x \, \mathcal{L}(x) \,, \tag{3.1}$$

[3]This restriction is essentially imposed by the requirement of locality.

we can get by with ordinary derivatives such as $\frac{\partial}{\partial\phi(x)}$.

To get accustomed to this setting, let us review the transition from the Lagrangian to the Hamiltonian formalism in the context of continuous degrees of freedom: let $\mathcal{L}$ be a function of a field ϕ and its derivatives. $\mathcal{H}$ is defined as its Legendre transform with regard to $\dot{\phi}$. We thus introduce the canonical conjugate field $\pi(x)$ via

$$\pi(x) := \frac{\partial\mathcal{L}(x)}{\partial\dot{\phi}(x)}\,, \tag{3.2}$$

solve this (algebraic!) equation for $\dot{\phi}(x)$,

$$\dot{\phi} = f(\phi, \nabla\phi, \pi)\,, \tag{3.3}$$

and then obtain $\mathcal{H}$ via

$$\mathcal{H}(\phi, \nabla\phi, \pi) = \pi\dot{\phi} - \mathcal{L}(\phi, \nabla\phi, \dot{\phi})\,. \tag{3.4}$$

Example 1. *Consider the following Lagrangian density for a real scalar field*

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \mathcal{V}(\phi)\,. \tag{3.5}$$

The canonically conjugate field is given by

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}(x)} = \dot{\phi}(x)\,. \tag{3.6}$$

Hence,

$$\begin{aligned} \mathcal{H} &= \pi\dot{\phi} - \mathcal{L} && (3.7)\\ &= \pi^2 - \frac{1}{2}\pi^2 + \frac{1}{2}\nabla\phi\cdot\nabla\phi + \mathcal{V}(\phi) && (3.8)\\ &= \frac{1}{2}\pi^2 + \frac{1}{2}\nabla\phi\cdot\nabla\phi + \mathcal{V}(\phi)\,. && (3.9) \end{aligned}$$

Make sure that you understand all signs and index placements.

The above example illustrates that Lorentz invariance is manifest in the Lagrangian density, but necessarily obscured in the Hamiltonian density (the Hamiltonian being the generator of time translations). Note furthermore that the analogue of the relations

$$L = K - V \quad H = K + V \tag{3.10}$$

familiar from classical mechanics hold in the above example if we set

$$\mathcal{K} = \frac{1}{2}\dot{\phi}^2\,. \tag{3.11}$$

In the following, it will however by more convenient to refer to all terms bilinear in the fields as kinetic terms, and refer to all other terms as interaction terms.

The Euler-Lagrange equations in the continuous setting take the form

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} - \frac{\partial \mathcal{L}}{\partial \phi} = 0\,. \tag{3.12}$$

They follow from imposing stationarity of the action

$$I = \int dt\, L = \int d^4x\, \mathcal{L}\,. \tag{3.13}$$

It is a good exercise to rederive them as such. You will have to assume that all fields and variations vanish (or fall off sufficiently quickly) at infinity, allowing you to drop total derivatives.

Example 2. *Continuing with the Lagrangian of example 1, we obtain the equations of motion*

$$\Box\phi + \mathcal{V}'(\phi) = 0\,, \tag{3.14}$$

where the prime on $\mathcal{V}$ *indicates differentiation with regard to* ϕ*. Setting* $\mathcal{V} = m^2\phi^2$*, we obtain the Klein-Gordon equation*

$$(\Box + m^2)\phi = 0\,. \tag{3.15}$$