ICFP M2 - Statistical physics 2
A reminder on some Gaussian identities

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- If $a > 0$ and $b \in \mathbb{C}$,

$$\int_{\mathbb{R}} \mathrm{d}x \, e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}} \ , \qquad \int_{\mathbb{R}} \mathrm{d}x \, e^{-\frac{1}{2}ax^2 + bx} = \sqrt{\frac{2\pi}{a}} e^{\frac{1}{2}\frac{b^2}{a}} \ .$$

- If A is an $n \times n$ real symmetric positive definite matrix and $\vec{b} \in \mathbb{C}^n$,

$$\int_{\mathbb{R}^n} \mathrm{d}\vec{x} \, e^{-\frac{1}{2}\vec{x}^T A \vec{x}} = \frac{(2\pi)^{n/2}}{\sqrt{\det A}} \ , \qquad \int_{\mathbb{R}^n} \mathrm{d}\vec{x} \, e^{-\frac{1}{2}\vec{x}^T A \vec{x} + \vec{b}^T \vec{x}} = \frac{(2\pi)^{n/2}}{\sqrt{\det A}} e^{\frac{1}{2}\vec{b}^T A^{-1} \vec{b}} \ ,$$

$$\int_{\mathbb{R}^n} \mathrm{d}\vec{x} \, x_i x_j e^{-\frac{1}{2}\vec{x}^T A \vec{x}} = \frac{(2\pi)^{n/2}}{\sqrt{\det A}} (A^{-1})_{ij} \ ,$$
$$\int_{\mathbb{R}^n} \mathrm{d}\vec{x} \, x_i x_j x_k x_l e^{-\frac{1}{2}\vec{x}^T A \vec{x}} = \frac{(2\pi)^{n/2}}{\sqrt{\det A}} [(A^{-1})_{ij}(A^{-1})_{kl} + (A^{-1})_{ik}(A^{-1})_{jl} + (A^{-1})_{il}(A^{-1})_{jk}] \ .$$

- One says that a random variable X is a Gaussian of mean μ and variance $\nu > 0$, to be denoted $X \overset{\mathrm{d}}{=} \mathcal{N}(\mu, \nu)$, if it has the density $f_X(x) = e^{-\frac{1}{2\nu}(x-\mu)^2} \frac{1}{\sqrt{2\pi\nu}}$. Then

$$\mathbb{E}[X] = \mu \ , \qquad \mathbb{E}[(X - \mu)^2] = \nu \ .$$

- One says that a vector of random variables $\vec{X} = (X_1, \dots, X_n)$ is a Gaussian of mean $\vec{\mu}$ and covariance matrix C (a real symmetric positive definite $n \times n$ matrix), to be denoted $X \overset{\mathrm{d}}{=} \mathcal{N}(\vec{\mu}, C)$, if it has the density $f_{\vec{X}}(\vec{x}) = e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T C^{-1} (\vec{x}-\vec{\mu})} \frac{1}{(2\pi)^{n/2}\sqrt{\det C}}$. Then

$$\mathbb{E}[X_i] = \mu_i \ , \qquad \mathbb{E}[(X_i - \mu_i)(X_j - \mu_j)] = C_{ij} \ .$$

- A random variable X is said to be centered if it has zero mean, $\mathbb{E}[X] = 0$.
- If X is a centered Gaussian random variable and $b \in \mathbb{C}$,

$$\mathbb{E}[e^{bX}] = e^{\frac{1}{2}b^2 \mathbb{E}[X^2]} \ .$$

- If $\vec{X}$ is a centered Gaussian random vector and $\vec{b} \in \mathbb{C}^n$,

$$\mathbb{E}\left[e^{\sum_{i=1}^n b_i X_i} \right] = e^{\frac{1}{2} \sum_{i,j=1}^n b_i b_j \mathbb{E}[X_i X_j]} \ .$$

- If X is a centered Gaussian random variable and F an arbitrary function (regular enough) from $\mathbb{R}$ to $\mathbb{R}$,

$$\mathbb{E}[X F(X)] = \mathbb{E}[X^2] \mathbb{E}[F'(X)] \ .$$

- If $\vec{X}$ is a centered Gaussian random variable and F an arbitrary function (regular enough) from $\mathbb{R}^n$ to $\mathbb{R}$,

$$\mathbb{E}[X_i F(X_1, \dots, X_n)] = \sum_{j=1}^n \mathbb{E}[X_i X_j] \mathbb{E}[(\partial_j F)(X_1, \dots, X_n)] \ .$$